Dynamical systems and chaos, MAT-35006

1. Discuss the SIRS-type model of infectious diseases: S=portion of the susceptible individuals in the population, I=infected ones, R=recovered ones, and a part of R may return to the S class ($S + I + R = 1$). Write this as a dynamical system in $\mathbb{R}^2$. Under what condition can the disease become permanently established in the community?

2. Consider the iterative mapping given by the family of functions

$$f_c(x) = x^2 + c, \quad x, c \in \mathbb{R}.$$

Discuss the existence and types of fixed points and the bifurcations that occur when c is decreased from $c > 1/4$ to $c = -3/4$.

3. Let us examine a perturbed Hamiltonian system

$$H(p, q) = H_0(p, q) + \varepsilon H_1(p, q),$$

where H_0 is an integrable Hamiltonian, H_1 is non-integrable, $p, q \in \mathbb{R}^2$, and $\varepsilon \in \mathbb{R}$ is small.

a. Describe the phase-space structure of the orbits of H, when $\varepsilon = 0$. What are action-angle variables? Write the above perturbation equation using them as arguments.

b. Explain how the phase-space structure changes when $\varepsilon > 0$.

c. What is the main difference between the chaos of a Hamiltonian system and that of a dissipative one (such as Lorenz)?

4.

a. What is the difference between the "center" fixed point of a linear dynamical system and the limit cycle attractor? Describe the condition(s) of the Hopf bifurcation.

b. Consider the Brusselator system:

$$\begin{cases} \dfrac{dX}{dt} = A - (B+1)X + X^2 Y \\ \\ \dfrac{dY}{dt} = BX - X^2 Y \end{cases}$$

Show that condition for the Hopf bifurcation in the system is $B = 1 + A^2$.

5. Consider the following chemical transformation:

$$A + A \xrightarrow{k} B$$

a. Write down the overall rate of the reaction as well as rates for each chemical species.

b. Speculate on possible limitations of using ODE approach when describing chemical kinetics in general and dynamics of this particular chemical reaction. How would you rewrite the reaction rate, if number of molecules A is too low?

6. There are four generic codim-1 bifurcations that any neuron undergoes when changing from "resting" to "spiking" state. Two of them are supercritical and subcritical Hopf bifurcations. Sketch the phase portraits on a plane before and after the bifurcation point in these two cases.

If you remember other two bifurcations, name them and/or sketch the phase portraits likewise.