Homework 2: Bifurcations + 2D linear systems

Bifurcations

1. Draw the bifurcation diagrams for the following equations depending on a parameter a:

   (a) $x' = x^2 + ax$

   (b) $x' = 2x^2 + x + a$

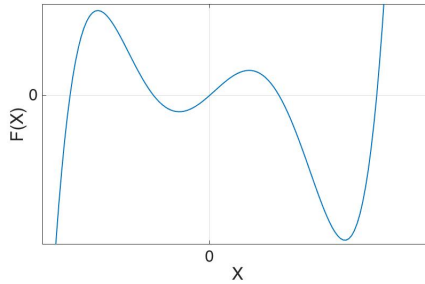   (c) $x' = x^3 - x^2 + a$

   (d) $x' = \cos x + a$

2. Consider the equation: $x' = ax - \sin x$

   (a) Explain the qualitative change in the phase line of the system using the graphs of the functions $f(x) = ax$ and $g(x) = \sin x$ as a changes from -1 to 1.

3. Consider the graph of a function $f(x)$:

   (a) Draw the bifurcation diagram of the equation $x' = g_a(x)$, where $g_a(x) = f(x) + a$ (**optional** exercise → extra points).

Linear planar systems

4. Suppose $A =$:

$$\text{I)} \begin{pmatrix} 2 & 1 \\ -5 & -4 \end{pmatrix} \quad \text{II)} \begin{pmatrix} -1 & 0 \\ 2 & -3 \end{pmatrix}$$

$$\text{III)} \begin{pmatrix} -1 & -1/2 \\ 9 & 2 \end{pmatrix} \quad \text{IV)} \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}$$

(a) Find eigenvalues and eigenvectors (in case of complex eigenvalues you may omit finding eigenvectors).

(b) Find the phase portraits below corresponding to the cases above, given the equations $X' = AX$

1)

2)

3)

4)
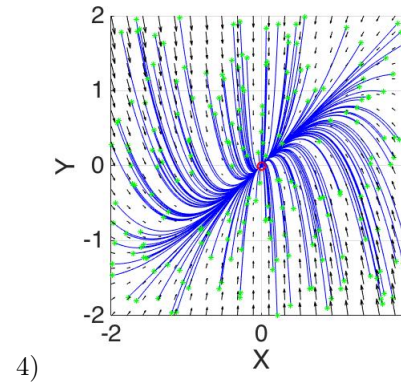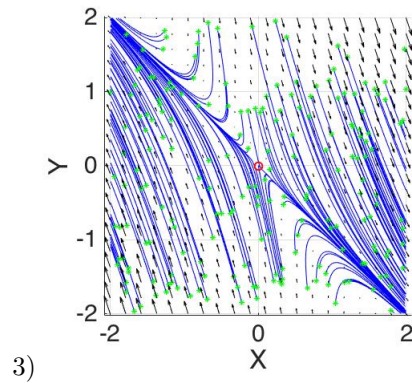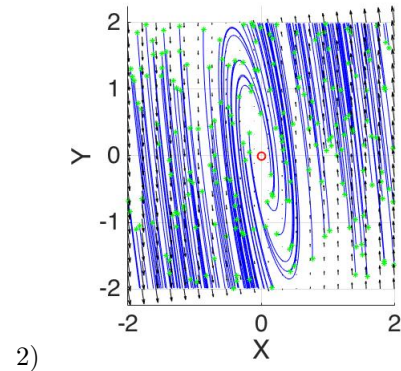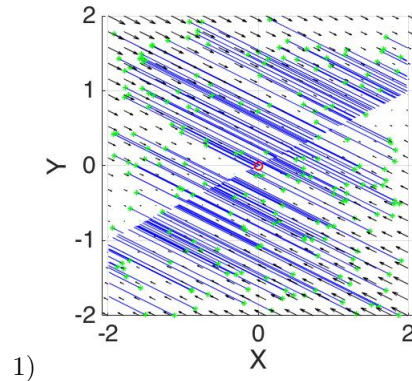
5. $X' = AX$, where $A = \begin{pmatrix} a & a \\ 2 & 2 \end{pmatrix}$

(a) Sketch the phase portrait of the system.

(b) Determine bifurcation values of a.

6. Consider the harmonic oscillator $x'' + x' + x = 0$:

(a) Sketch the phase portrait.

(b) What is the period of the oscillator? (**optional**)