Homework 4: Discrete systems + chaos

Find in the enclosed material Matlab function codes for `run_map.m`, `lorenz.m`, `lyapunov.m`

1. Consider the map $x_{n+1} = x_n^2 - 1$.

   (a) Find the steady states of the system.

   (b) Does the system have stable equilibria? Support your conclusion with calculations.

   (c) Simulate the system using `run_map.m` function: i) write a function handle in Matlab corresponding to the function $f(x) = x^2 - 1$, ii) take $x_0 = 0.5$ and `NI`$= 40$, then iii) simulate with `run_map(f,`x_0`,NI);`.

   (d) What dynamical regime do you see from the simulation?

   (e) Use the plots of $f^n(x)$ and $y = x$ to show whether the system has 2-, 3-, 4-, or 5-cycle regimes. Draw your conclusions from the plots.

2. Bifurcations do occur in the discrete maps as well. The bifurcation criteria are $f'(\bar{x}) = \pm 1$, since the steady state loses/gains its stability. Consider the map $f(x) = ax - 1$.

   (a) Find the steady states of the system.

   (b) Find the bifurcation criteria for the parameter a.

   (c) Simulate the system for $a = -1$ and different initial conditions x_0 using `run_map()` function. What do you see?

   (d) What would you expect for $a < -1$? Do the simulations support your expectations? $f'(\bar{x}) = -1$ bifurcation condition usually gives 2-cycles (period doubling bifurcation), why do you think you cannot see the period-n trajectories for this map (*hint*: see the form of $f^n(x)$ functions)?

   (e) What would you expect for $a = 1$ and $a > 1$? Can you see that in the simulations?

3. Consider the logistic map $f(x) = \lambda x(1-x)$. You need to calculate and plot the bifurcation diagram of this map for $\lambda \in [2.5, 4.0]$ with step $d\lambda = 0.01$.

   (a) In Matlab insert the λ vector, like `lam = 2.5:.01:4.0;`

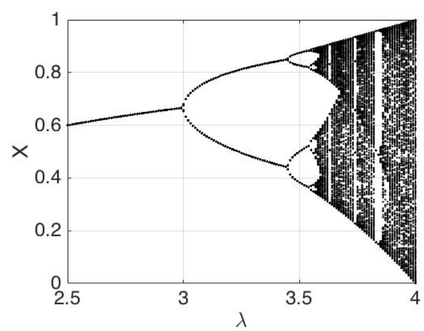
(b) For each of the values in `lam` (use `for` or other vectorized techniques) you need to simulate the trajectory using `run_map()`.

(c) First simulate the initial portion of the trajectory like `x = run_map(f,0.1,300);` to be sure that you are at the attractor.

(d) Then start another iteration just from the end point of the previous simulation: `run_map(f,x(end),300);`. Save this last time series for the later processing (*hint:* use cell arrays or 3D matrices for each trajectory). These time series you will need in the last exercise, so you may want to keep them safe on the disk or in the workspace.

(e) Plot each point of the stored trajectories against the corresponding λ value in `lam` (*hint*: plot with dots for nicer view, i.e. `'.'` option to `plot` command).

(f) You should see something like this:



4. Consider the Lorenz system:

$$\begin{cases} x' = \sigma(y - x) \\ y' = rx - y - xz \\ z' = xy - bz \end{cases}$$

(a) Find the steady states of the system.

(b) Write down the Jacobian of the linearized system.

(c) Simulate the system with `lorenz(sig,r,b)`, where `sig = 10`, `r = 28`, and `b = 8/3`.

(d) For the given parameter set calculate the actual values of the steady states and plot them along with the Lorenz attractor (which you must have gotten as a result of `lorenz()` simulation). Plot them as big red dots or other color to see them clearly against the black Lorenz attractor (*hint*: use `'MarkerSize'` option to `plot`). Can you see that the system rotates around and between the steady states never reaching them as they are unstable?

(e) Simulate the lorenz system with slightly different initial conditions, say, `x0 = [1 3 5]` and `x1 = [1.005 3 5]` using `[t0,y0] = lorenz(10,28,8/3,x0);` and `[t1,y1] = lorenz(10,28,8/3,x1);`. Plot the corresponding variables vs. time for the two simulations against each other. Do you see the *sensitivity* property? How would you describe it?

5. Return to the logistic map and the trajectories you saved for different λ values, when you plotted the diagram.

   (a) For each of the trajectory calculuate Maximal Lyapunov Exponent (MLE) using `lyapunov()` function in Matlab: `lyapunov(x,dt)`, where `x` is the saved trajectory and `dt = 1` is a time step.

   (b) Plot MLE for each λ value. It would be even better, if you could align this plot with the logistic diagram you plotted before to get something like this: