Homework 6: Population dynamics

1. For the Verhulst equation $\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)$:
   (a) Determine steady states.
   (b) Can the found steady states coexist, i.e. be stable for the same parameter set?

2. The spread of infectious diseases can be modeled by a system of nonlinear ODE's. The simplest known example is so called SIR model. The total population is divided into three disjoint groups: Susceptible (S), Infected (I), and Recovered (R) individuals. The model assumes for simplicity that the total population is constant, i.e. $(S+I+R)' = 0$. Susceptible individuals become infected when the two meet and infected ones recover at certain rate, i.e. $S' = -\beta SI$ and $I' = \beta SI - \nu I$ (R can be determined once $S(t)$ and $I(t)$ are found given the constant population size).
   (a) Find equilibria and S- and I-nullclines.
   (b) Sketch the phase portrait of the system.
   (c) The constant size of the population suggests that there can be a constant of motion. Try to identify it, using the analogy of the Volterra equations from the lecture.
   (d) Plot the levels of the constant of motion in Matlab to verify you sketched the phase portrait correctly.

3. Consider the map (applied to real populations): $N_{t+1} = \frac{rN_t}{1 + bN_t^2}$, $b > 0$ (you may want to use `run_map` to support your calculations).
   (a) Identify the steady states and determine parameter conditions of their existence.
   (b) For each steady state find parameter conditions of its stability.
   (c) Identify the bifurcation values of the parameter r, where stability of the steady states or their number change.
   (d) What is the biological meaning of the parameter r (recall logistic equation)? Can it be negative?

(e) Find the cycles of the system for $b = 1$. If you have identified any cycle, can it occur in real situation according to the model? Explain why?

4. (**extra/optional**) Recall Kolmogorov system:

$$\begin{cases} \dfrac{dx}{dt} & = & k_1(x)x - L(x)y \\ \dfrac{dy}{dt} & = & k_2(x)y \end{cases}$$

Steady states are:

$$\text{i)} \begin{cases} x = 0 \\ y = 0 \end{cases} \qquad \text{ii)} \begin{cases} x = A \\ y = 0 \end{cases} \qquad \text{iii)} \begin{cases} x = B \\ y = C \end{cases},$$

where $k_1(A) = 0$, $k_2(B) = 0$, and $C = \dfrac{k_1(B)B}{L(B)}$.

(a) Remembering that $\dfrac{dk_1(x)}{dx} < 0$, $\dfrac{dk_2(x)}{dx} > 0$, $k_2(0) < 0 < k_2(\infty)$, $L(x) > 0$, prove that i) is saddle, ii) is a stable node/saddle (determine conditions), and iii) is stable/unstable node or focus (determine conditions).