Dynamical Systems and Chaos
Part I: Theoretical Techniques

Lecture 1: 1D analysis

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Preliminary notes

- Semi-lecture mode: some elements of a seminar, e.g homework revision.
- Prerequisite: knowledge of Differential Equations.
- Home page for the course: `http://www.cs.tut.fi/~potapov/dyn_syst_chaos/index.html`
- Requirement: Homeworks (see details on the web page).
- Optional: Project (successful project increases your score by 1 on the exam).
- The possible projects are listed on the web-site of the course. Not all the projects are biology driven (not all students are inclined toward biology).
- The first part of the course — theory, the second — applications (main theme is Biology).

Dynamical system

Dynamical systems imply:

- Movement over **time**, or evolution (however, "evolution" assumes some progress as well).
- Space where the system's movement is realized: **phase space** – all possible states that the system can (potentially) occupy.
- The movement over time in the phase space is governed by the **law of evolution**. In general, the dynamical systems theory assumes the deterministic operator of evolution.

For example:

- x is a state variable (then, the phase space is a line)
- $r \cdot x$ (r is a constant) defines change of x over time, i.e. derivative of x, and hence the law of evolution is $dx/dt = r \cdot x$.

Simplest example

$$\frac{dx}{dt} = r \cdot x \Rightarrow \frac{dx}{x} = r \cdot dt \Rightarrow \int \frac{dx}{x} = \int r \cdot dt$$

$$\Rightarrow \ln x = r \cdot t + C \Rightarrow x(t) = C_0 \cdot \mathrm{e}^{r \cdot t}$$

C is an integrational constant made in the form of $C = \ln C_0$ for better representation.
So, $C_0 \equiv \mathrm{e}^C$ is a constant determined from the initial conditions: $t = 0 : x(0) \equiv x_0 = C_0 \cdot \mathrm{e}^{r \cdot 0} \Rightarrow x_0 = C_0$
So, the solution of the simplest example:

$$x(t) = C_0 \cdot \mathrm{e}^{rt}\,, C_0 = x_0 \equiv x(t = 0)$$

Solution plotted

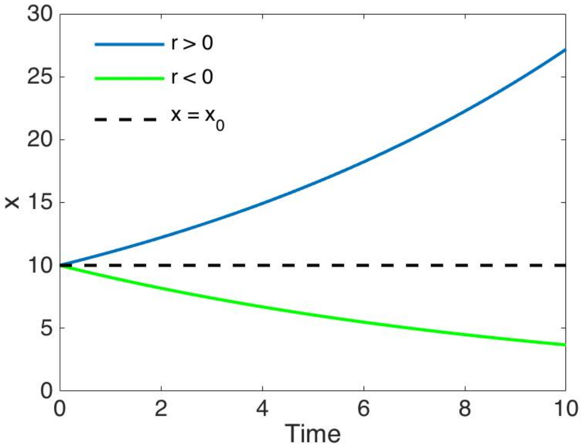


Figure: Solution of the simplest example $dx/dt = r \cdot x$ for the single initial condition $x = x_0 \equiv 10$ and fixed $r = \pm 0.1$.

Parameter r

- r is a parameter of the equation $dx/dt = r \cdot x$
- If $r > 0$ the solution $x(t) = C_0 \cdot \mathrm{e}^{r \cdot t}$ goes to $+\infty$ with time going to $+\infty$ (see the Figure), given $C_0 > 0$ and $x(t) \to -\infty$, given $C_0 < 0$.
- If $r < 0$ the solution $x(t)$ goes to zero in forward time for all C_0.

There are two different behaviours, when r is positive and negative. So, the boundary value $r = 0$, when $x(t)$ does not change and $x = C_0$ for all times, is so called *bifurcation value* demarcating the frontier between parameter regions with two distinct dynamical behaviours.

Steady state

- **Steady state** is the value of x making the derivative $dx/dt = 0$.
- Steady states are the resting points of the system.
- Although $r = 0$ makes $dx/dt = 0$ for any x, it is a trivial solution.
- For $r \neq 0$ the only $\bar{x} = 0$ makes the system resting (not moving).
- It can be seen from: $dx/dt = 0 \Rightarrow r \cdot x = 0 \Rightarrow x = 0$, given $r \neq 0$.
- *Steady states are one of the most important solutions of the differential equations.*

Steady state and equilibrium

- "Steady state" and "equilibrium" bear almost similar connotations, especially, in mathematics.
- However, there is a significant difference, when used in physics or biology.
- Steady state refers only to the resting states of the variables of the system, in other words, states where $dx/dt = 0$ (variables do not change in time).
- Equilibrium refers to a state where energy (Gibbs energy, including entropy) is equal zero.
- Non-equilibrium steady states are possible, where, although $dx/dt = 0$, entropy or other flows are non-zero.
- Thus, steady state is a broader and more mathematical notion, while (non)equilibrium implies some physical properties of the system.
- Example: biological cell might be in a steady state growth phase, but is away from the equilibrium (the equilibrium would mean death for the cell).

Stable and unstable steady states

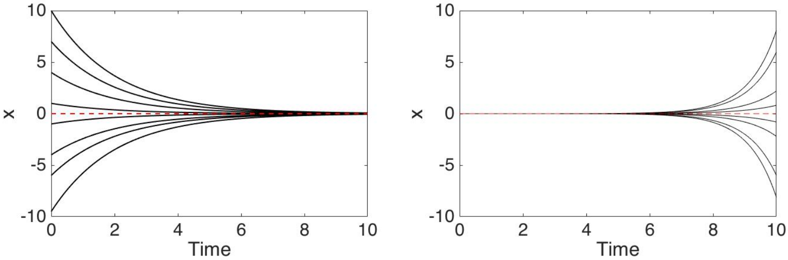


Figure: Negative r (left) and positive r (right) make the same steady state $\bar{x} = 0$ *stable* and *unstable*, respectively.

Analytically, $C_0 \equiv x(0) = 0$ would let the system rest at the steady state even with positive r, because $dx/dt = 0$ for the initial value does not allow system to move away from the equilibrium. But practically (in computer simulations) it is rarely the case. Why?

Stability of the steady states

- **Stable** steady state is called **sink** since the trajectory (i.e. function $x(t)$) tends toward it in forward time.
- **Unstable** steady state is called **source** since the trajectory $x(t)$ tends toward it in backward time (or tends away from it in forward time).
- **Saddle** point is neither sink nor source. However, saddle point is unstable.

Which system is stable?

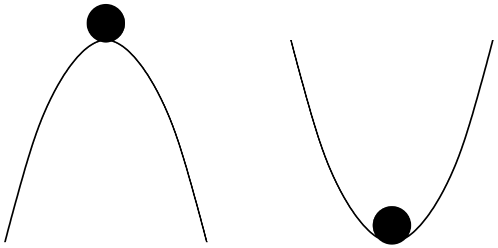
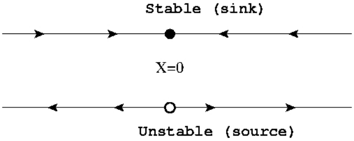


Figure: The notion of stability according to Lyapunov. Ball at the top of the hill is unstable, but in the valley it is stable.

Phase space representation

- Sometimes it is useful to exclude time from the graphical representation.
- In this case we talk about *phase space*.
- Phase space has dimension equal to the dimension of the system (without time), thus, for the simplest example $dx/dt = r \cdot x$, we have a phase line.
- Single point moving in the phase space represents the whole system evolving over time.



- $\bar{x} = 0$ is an equilibrium.
- The system tends toward the sink.
- The system tends away from the source.
- The tendencies are shown with arrows.

Multiple equilibria (1D)

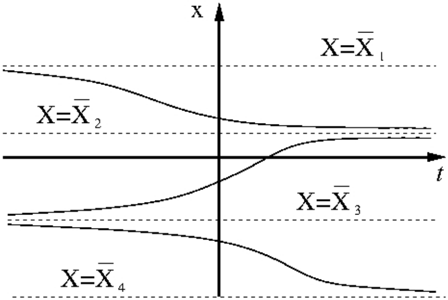


Figure: Multiple equilibria $\bar{x}_i$ with solutions $x(t)$. $\bar{x}_{\{2,4\}}$ are stable, $\bar{x}_{\{1,3\}}$ are unstable.

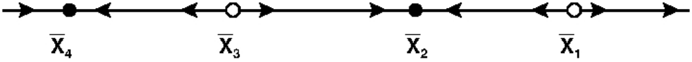


Figure: Phase line representation of the above figure. Empty/filled circles denote unstable/stable solutions.

How to find equilibria?

- Consider general form ODE:

$$\frac{dx}{dt} = f(x)$$

- $f(x)$ is a right-hand side of the equation (RHS).
- Steady state (resting point) implies $dx/dt = 0$, that is:

$$f(x) = 0$$

- $f(x) = 0$ is an algebraic equation. The solutions to the equation give steady states $\bar{x}$ of the ODE above.
- Moreover, $\bar{x}$ equilibria divide the phase line into regions, where $f(x)$ has to be either positive or negative, which defines the tendencies of the ODE solutions $x(t)$. Namely:
- If $f(x) > 0$, then $x(t) \to +\infty$
- If $f(x) < 0$, then $x(t) \to -\infty$

Multiple equilibria example

- Consider: $dx/dt = x^3 - 9x$
- $f(x) = x^3 - 9x$.
- $f(x) = 0 \Rightarrow x^3 - 9x = 0 \Rightarrow x(x^2 - 9) = 0 \Rightarrow x(x-3)(x+3) = 0 \Rightarrow$

$$\left[\begin{array}{l} \bar{x}_1 = -3 \\ \bar{x}_2 = 0 \\ \bar{x}_3 = 3 \end{array}\right.$$

- Determine sign of $f(x)$ in each sub-region:

$$\left[\begin{array}{r} x < \bar{x}_1 : f(x) < 0 \\ \bar{x}_1 < x < \bar{x}_2 : f(x) > 0 \\ \bar{x}_2 < x < \bar{x}_3 : f(x) < 0 \\ x < \bar{x}_3 : f(x) > 0 \end{array}\right.$$

Multiple equilibria example (continued)

- In regions with $f(x) < 0$ $x(t)$ tends to $-\infty$, in regions with $f(x) > 0$ $x(t)$ tends to $+\infty$.
- Note: only sign of $f(x)$ is important, not the actual form.
- From the figure below, we can determine which steady states are stable and which are not: $\bar{x}_1 = -3$ (unstable), $\bar{x}_2 = 0$ (stable), $\bar{x}_3 = 3$ (unstable).

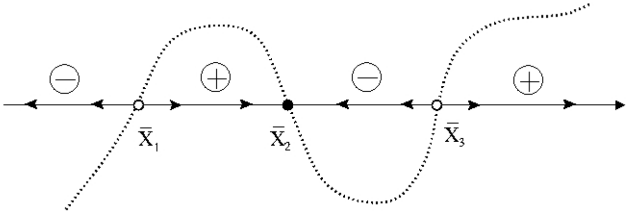


Figure: Regions of the phase line with different signs of $f(x)$ with a shematic of the function.

Saddle

- Consider equation: $dx/dt = x^4 - x^2$
- $f(x) = 0 \Rightarrow x^2(x^2 - 1) = 0 \Rightarrow$

$$\left[\begin{array}{l} \bar{x}_1 = -1 \\ \bar{x}_2 = 0 \\ \bar{x}_3 = 1 \end{array}\right.$$

- There are system's solutions $x(t)$ that tend toward the saddle point and, at the same time, there are solutions that tend away from the saddle point.

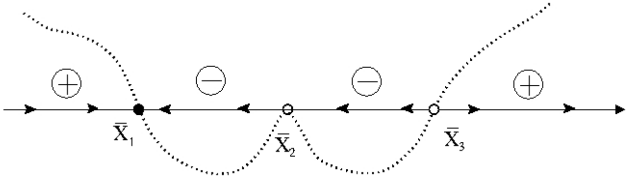


Figure: $\bar{x}_2 = 0$ is a saddle point. Saddle points are unstable.

Patterns of RHS

- Have you noticed patterns of $f(x)$ around the steady states of each particular type?

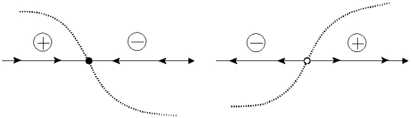


Figure: Stable (left) and unstable (right) patterns.

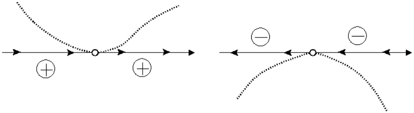


Figure: Saddle patterns.

- $f(x)$ changes sign in certain order around steady state $\bar{x}$, thus, it is connected to the sign of the derivative $f'(\bar{x})$
- $f'(\bar{x}) > 0 \Rightarrow \bar{x}$ is unstable.
- $f'(\bar{x}) < 0 \Rightarrow \bar{x}$ is stable.
- $f'(\bar{x}) = 0 \Rightarrow \bar{x}$ is a saddle.

Summary

- We have considered simple and effective pen-and-paper techniques to analyze the dynamical behavior of 1D systems.
- Phase space/line(1D) analysis is proven to be higly productive method for qualitative assessments on dynamics.

Useful resources

- The main source for the course:
  M. Hirsch, S. Smale & R. Devaney: Differential Equations, Dynamical Systems and an Introduction to Chaos, ISBN 0123497035, Academic Press 2002.
  Use Chapter 1 to revise the content of this lecture.