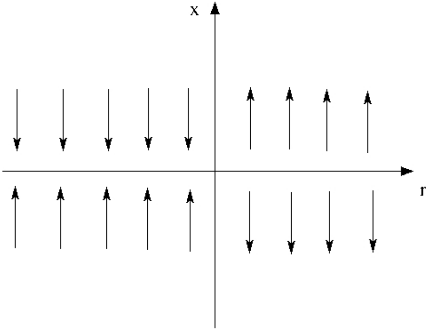
Dynamical Systems and Chaos
Part I: Theoretical Techniques

Lecture 2: Bifurcations + 2D linear systems

Ilya Potapov
Mathematics Department, TUT
Room TD325

The simplest example revisited

- $dx/dt = r \cdot x$.
- $\bar{x} = 0$ is a steady state.
- $r < 0$: $\bar{x}$ is stable.
- $r > 0$: $\bar{x}$ is unstable.
- $r = 0$: singularity point, $x = 0$ always (not moving in time).



- Solutions as a function of parameters for the "simplest example".
- Any vertical cross-section through the plane is a phase line.
- Arrows denote movement of the system along the phase lines.

The extended simplest example

- $dx/dt = r \cdot x + a$
- $a > 0$ and is a constant.

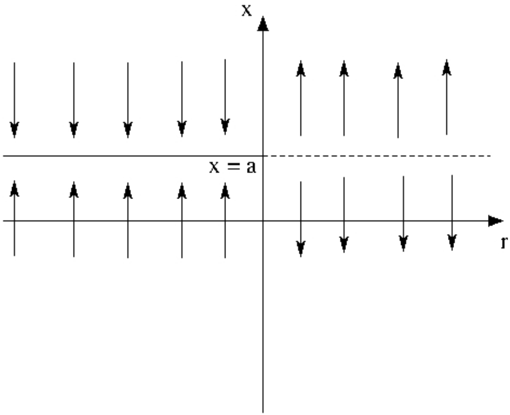


Figure: Note we have introduced new notations: *solid* line indicates stable steady state, whereas *dashed* line — unstable steady state.

Bifurcations of Dynamical Systems

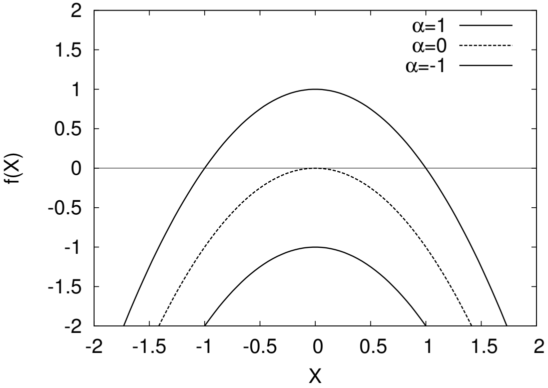
- *Bifurcation* is a significant/drastic change in the dynamical behaviour of the dynamical system...
- ... or, in other words, change in topology of the phase space.
- Bifurcations take place, when parameters of the system vary.
- Parameter value, at which a bifurcation occurs, is called *bifurcation value.*
- *Bifurcation diagram* is a plot showing solutions of the system as a function of parameter(s).
- The number of parameters needed to be changed for a bifurcation to occur is called *codimension* (codim) of the bifurcation.

Simple codim-1 bifurcation

Consider the system

$$\dot{x} = f(x) \equiv \alpha - x^2$$

For different α system has different number of steady state solutions and the phase portrait of the system changes.



Bifurcation diagram I

- $\dot{x} = \alpha - x^2$
- Steady states are: $\alpha = x^2$.

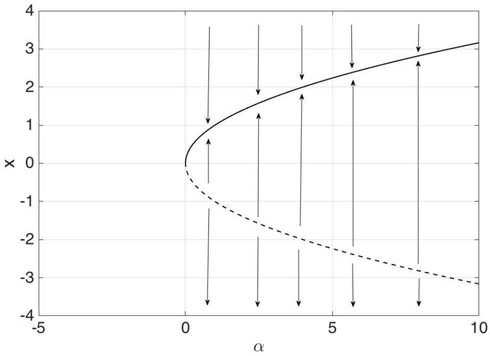
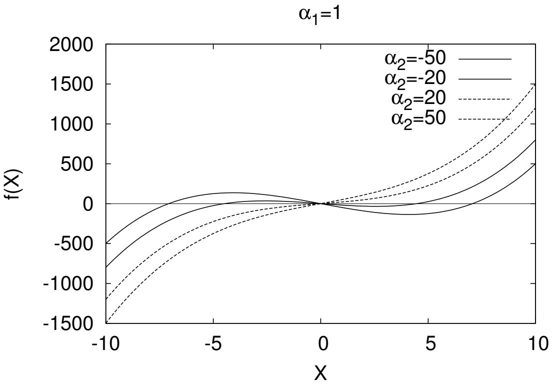


Figure: Bifuraction diagram: two steady states exist only for positive α values, while they coalesce at $\alpha = 0$ and disappear, i.e. there are no equilibria for $\alpha < 0$.

Codim-2 bifurcation: cusp bifurcation

$$\dot{x} = \alpha_1 + \alpha_2 x + x^3$$

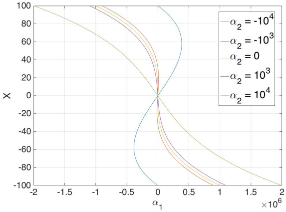
For negative and positive α_2 the system has considerably different phase portraits.

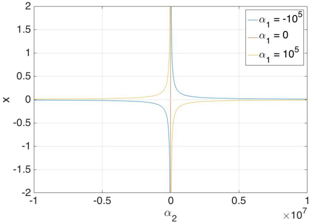


Bifurcation diagram II

- $\dot{x} = \alpha_1 + \alpha_2 x + x^3$
- Steady states are: $\alpha_1 = -x^3 - \alpha_2 x$, $\alpha_2 = \dfrac{-x^3 - \alpha_1}{x}$
- The bifurcation diagram is in 3D: (α_1, α_2, x).

Projections:





- NOTE: $\alpha_1 = 0$ (right) gives a parabola (see the steady state relations).

Bifurcation diagram II

- $\dot{x} = \alpha_1 + \alpha_2 x + x^3$
- Bistability region is between curves LP_1 and LP_2: two sinks separated by a saddle.
- After **cusp** there is only one stable steady state.

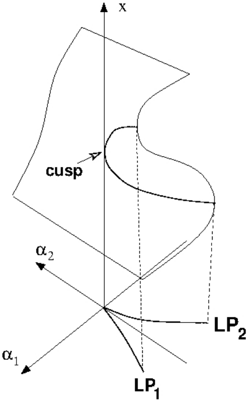


Figure: Bistability between LP_1 and LP_2 curves (Left) and monostable dynamics everywhere else (Right).

Planar linear systems (2D)

- We consider a system of the form:

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

- We will often use the matrix representation:

$$X' = AX$$

- that is

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- In this case $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix of the coefficients.

Planar linear systems: example

- Consider an example:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x \end{cases} \Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- At every time moment t the tangent vector to the point $(x(t), y(t)))$ is determined by the RHS. That is, the tangent vector is $(y(t), -x(t))$.

Planar example (continued)

- $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

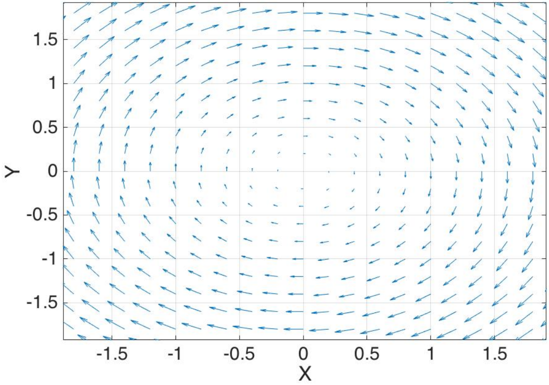


Figure: The solution $(x(t), y(t))$ of the system winds its way on the (x, y)-plane along the arrows, which are the numerical approximations of the tangent vectors.

Connection with 2nd order ODE's

- Planar (2D) linear systems are usually connected with linear 2nd order ODE's, which are important in physics.
- For example, the *harmonic oscillator* equation is:

$$mx'' + bx' + kx = 0$$

- It can be rewritten in the form:

$$\begin{cases} x' = y \\ y' = -\dfrac{b}{m}y - \dfrac{k}{m}x \end{cases}$$

- Or:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

2D equilibria

- $X' = AX$.
- In order to find equilibria we need to solve algebraic system:
$$\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$$
- The equations correspond to lines passing through the origin $(0, 0)$. The intersection point between the two lines is an equilibrium point. The intersection is always $(0, 0)$.
- The system has: i) a single unique equilibrium point $(0, 0)$, or ii) it has infinite number of solutions.
- case ii) takes place, when lines corresponding to each of the equations are the same.
- **From linear algebra:** $\det A \neq 0 \Rightarrow$ **case i,** $\det A = 0 \Rightarrow$ **case ii.**

Note from linear algebra

- For the system

$$\begin{cases} ax + by = \alpha \\ cx + dy = \beta \end{cases} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

  the following is true (given A is non-zero matrix):
- $\det A \neq 0$: unique solution (intersection of two lines).
- $\det A = 0$:
  - inifinite number of solutions (the two lines coincide).
  - no solutions (the lines are parallel).
- $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: the two lines pass through the origin $(0, 0)$ and cannot be parallel (when they are parallel they coincide).

Solutions of the 2D systems

- $X' = AX$
- The function $X(t) = e^{\lambda t} V_0$ is a solution, where V_0 is a eigenvector, i.e. it satisfies $AV_0 = \lambda V_0$, where λ is called eigenvalue and $\lambda \in \mathbb{R}$.
- Let us compute

$$\begin{aligned} X'(t) &= \lambda e^{\lambda t} V_0 \\ &= e^{\lambda t}(\lambda V_0) \\ &= e^{\lambda t}(AV_0) \\ &= A(e^{\lambda t} V_0) \\ &= AX(t) \end{aligned}$$

- Thus, the solution $X(t)$ is bound with the eigenvector.

Eigenvectors and eigenvalues

To find the eigenvector we need to solve:

$$AV_0 = \lambda V_0 \Rightarrow A\begin{pmatrix}x\\y\end{pmatrix} = \lambda\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow A\begin{pmatrix}x\\y\end{pmatrix} - \lambda\begin{pmatrix}x\\y\end{pmatrix} = 0$$

$$\Rightarrow (A - \lambda I)\begin{pmatrix}x\\y\end{pmatrix} = 0$$

- $I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ is an identity matrix.
- $(A - \lambda I)\begin{pmatrix}x\\y\end{pmatrix} = 0$ is also a system of linear equations regarding x and y.
- But now (!) we are interested in **nonzero/nonunique** solutions, because x and y are the components of the eigenvector V_0.
- Thus, $\det(A - \lambda I) = 0$ is a condition.

Eigenvalues

- $\det(A - \lambda I) = 0$ is equivalent to $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$, given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- This leads to a quadratic equation in λ (*characteristic equation*):

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \Rightarrow (a-\lambda)(d-\lambda)-bc = 0 \Rightarrow \lambda^2-(a+d)\lambda+ad-bc$$

- Finally:

$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

- Then we can find the eigenvectors associated with each of λ_i, e.g. $V_i = \left(\frac{b}{\lambda_i - a} \cdot y, y\right)$ or $V_i = \left(x, \frac{c}{\lambda_i - d} \cdot x\right)$ (note system is redundant).

Time-dependent solutions

- We have shown that $X_i(t) = \mathrm{e}^{\lambda_i t} V_i$ (V_i is an associated eigenvector) is a *straight-line* solution of the system. But this is not a general solution for any initial value.
- It can be shown (see Hirsch et al. book) that the general solution is:

$$Z(t) = C_1 \mathrm{e}^{\lambda_1 t} V_1 + C_2 \mathrm{e}^{\lambda_2 t} V_2 \equiv C_1 X_1(t) + C_2 X_2(t)$$

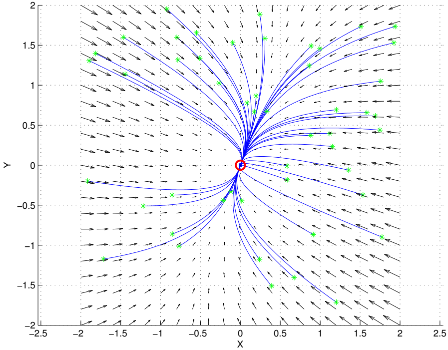
- Generally, if $X_1(t)$ and $X_2(t)$ are solutions, then it can be proven that their linear combination $C_1 X_1(t) + C_2 X_2(t)$ is also solution. Then it proves that this form is a unique solution for any initial value $Z(0) = C_1 X_1(0) + C_2 X_2(0)$.
- Constants C_1 and C_2 are to be found from the initial conditions.

Temporal behavior of solutions

- V_1 and V_2 are linearly independent vectors and, thus, can form a basis in $\mathbb{R}^2$ (that is, any other vector can be expressed as a linear combination of these two).
- So the temporal movement of the general solution $Z(t) = C_1 e^{\lambda_1 t} V_1 + C_2 e^{\lambda_2 t} V_2$ will be determined by $\lambda_{1,2}$ as to how *fast* ($|\lambda_i|$ value) and *toward or from* the origin (sign of λ_i) the system moves along the correpsponding vectors V_i.

λ_1, λ_2 — both negative (SINK)

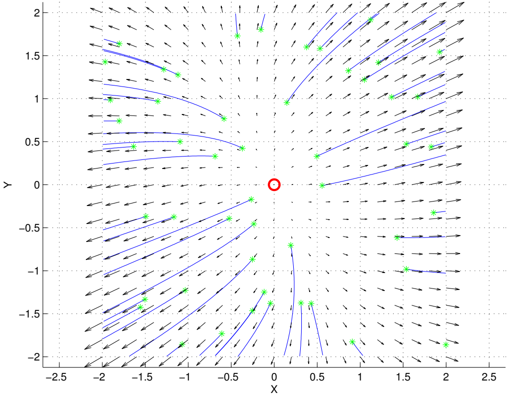
$\operatorname{Re}\lambda_{1,2} < 0 \Rightarrow$ both $X_1(t)$ and $X_2(t)$ components tend to zero.
For example, $a = -10$, $b = 2$, $c = 2$, $d = -5 \Rightarrow$
$\lambda_{1,2} = \frac{-15 \pm \sqrt{225 - 4\cdot(50-4)}}{2} = -7.5 \pm \frac{\sqrt{41}}{2}$ ($\sqrt{41} \approx 6.4$)



Here, the blue lines — solutions, green points — initial conditions, red circle — the equilibrium, arrows — vectors tangent to the solutions.

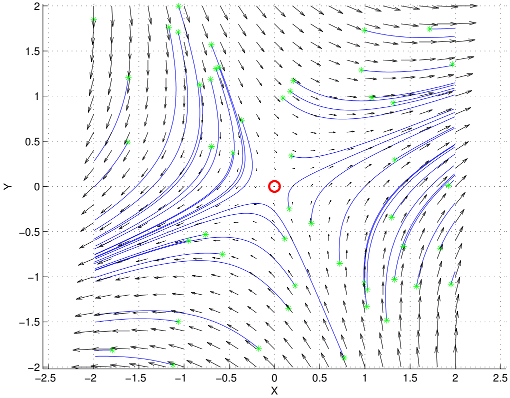
λ_1, λ_2 — both positive (SOURCE)

$\mathrm{Re}\,\lambda_{1,2} > 0 \Rightarrow$ both $X_1(t)$ and $X_2(t)$ components tend to ∞.
For example, $a = 10$, $b = 2$, $c = 2$, $d = 5 \Rightarrow$
$\lambda_{1,2} = \frac{15 \pm \sqrt{225 - 4\cdot(50-4)}}{2} = 7.5 \pm \frac{\sqrt{41}}{2}$ $(\sqrt{41} \approx 6.4)$



λ_1, λ_2 — different signs (SADDLE)

$\operatorname{Re}\lambda_1 < 0, \operatorname{Re}\lambda_2 > 0 \Rightarrow$ component $X_1(t)$ tends to zero whereas $X_2(t)$ tends to ∞. For example, $a = 1$, $b = 1$, $c = 1$, $d = -1 \Rightarrow$
$\lambda_{1,2} = \frac{0 \pm \sqrt{0^2 - 4\cdot(-1-1)}}{2} = \pm\sqrt{2}$



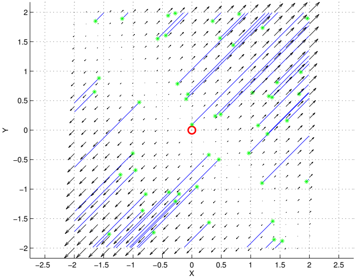
One λ equals zero (REDUNDANT)

That is, solutions: i) do not move along the vector corresponding to the zero λ and ii) tend away (the nonzero $\lambda > 0$) or toward (the nonzero $\lambda < 0$) an equilibrium along the vector corresponding to the nonzero λ. Thus, all points of the line corresponding to the zero-λ vector become equilibria.

For example, $a = 1$, $b = 1$, $c = 1$, $d = 1 \Rightarrow$

$\lambda_{1,2} = \frac{2 \pm \sqrt{2^2 - 4 \cdot (1-1))}}{2} = 0 \vee 2$. In this case,

$V_1 = (-1, 1)$ (equilibrium line) and $V_2 = (1, 1)$ (parallel to the solutions).

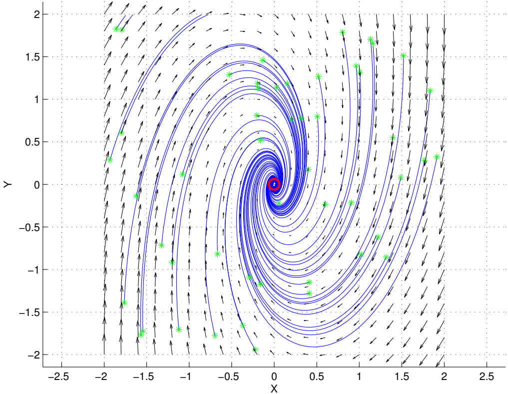


NOTE: the program drew only one default equilibrium point $(0, 0)$, although in this case the whole line is equilibria.

λ_1, λ_2 — complex conjugates (FOCUS/SPIRAL)

$\lambda_{1,2}$ is of the form $a \pm bi$. For example, $a = -1$, $b = 1$, $c = -4$, $d = -1 \Rightarrow \lambda_{1,2} = \frac{-2 \pm \sqrt{(-2)^2 - 4\cdot(1-(-4))}}{2} = -1 \pm 2i$

The system has oscillatory solution (complex eigenvalues), but with damped ($\mathrm{Re}\,\lambda_{1,2} < 0$) or expanding ($\mathrm{Re}\,\lambda_{1,2} > 0$) amplitude. The former is *stable* focus, the latter is *unstable* focus.

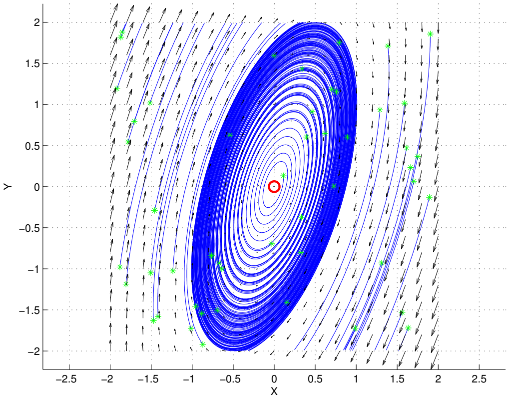


λ_1, λ_2 — complex conjugates. $\mathrm{Re}\,\lambda_{1,2} = 0$ (CENTER)

Special case of the focus point, so called `center` $\Rightarrow$ oscillatory regime.

For example, $a = 1$, $b = 1$, $c = -4$, $d = -1$ $\Rightarrow$

$\lambda_{1,2} = \frac{0 \pm \sqrt{0^2 - 4\cdot(-1-(-4))}}{2} = \pm\sqrt{3}i$



Classification of planar systems

For the system of equations:

$$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases} \overset{\text{vectorized}}{\Rightarrow} \frac{d\mathbf{X}}{dt} = \mathbf{AX}$$

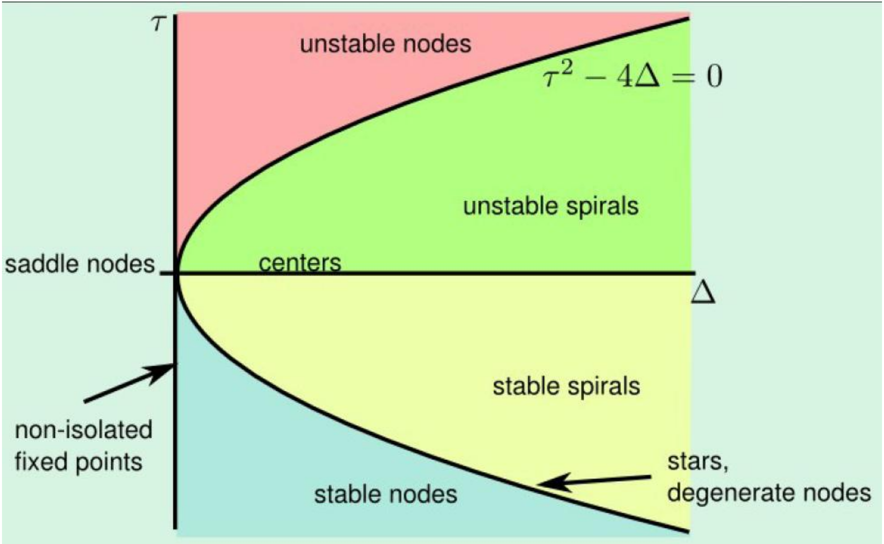
The characteristic polynomial is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, i.e. for 2D:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \tau\lambda + \Delta = 0\,,$$

where $\tau = a + d$ and $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ are *trace* and *determinant* of the matrix $\mathbf{A}$, respectively. NOTE: $\tau = \lambda_1 + \lambda_2$ and $\Delta = \lambda_1\lambda_2$.

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

Solutions plane



Wikipedia: `http://en.wikipedia.org/wiki/Linear_dynamical_system`

Summary

- Linear systems are easy to analyse.
- Moreover, linear systems analysis is a powerful tool to understand qualitative dynamics of the nonlinear systems as well, which we will see in later lectures.
- To revise the material see Hirsch et al. book, Chapters 2–4.