Dynamical Systems and Chaos
Part I: Theoretical Techniques

Lecture 3: Nonlinear systems

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Nonlinear systems

$$\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} F_1(x_1, x_2, \dots, x_n) \\ F_2(x_1, x_2, \dots, x_n) \\ \dots \\ F_n(x_1, x_2, \dots, x_n) \end{pmatrix}$$

- $F_i(\mathbf{x})$ are generally defined nonlinear functions (non-autonomous case, i.e. no time).
- As a general rule: no explicit analytical solution.
- Hence, there is a need for:
  - Numerical methods (a numerically calculated solution for any particular initial condition).
  - Various analytical techniques (topological, analytic, geometric).
- Moreover, particular solutions may be worthless (chaos/randomness), since do not show the overall dynamics of the system.

Planar systems

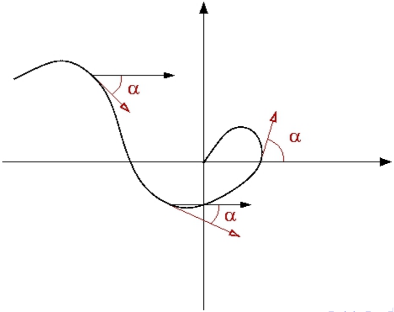
$$\begin{cases} \dfrac{dx}{dt} = P(x,y) \\ \dfrac{dy}{dt} = Q(x,y) \end{cases}$$

- We consider 2D systems as they are easier to analyse (algebraically, geometrically).
- Thus, we have a *phase plane* (cf. phase line for 1D).
- Since geometrically plane gives larger shape variants for the trajectories we talk about *phase portraits* of systems (in 1D there were no corresponding notion, but we saw phase portraits in 2D linear systems).

Vector field

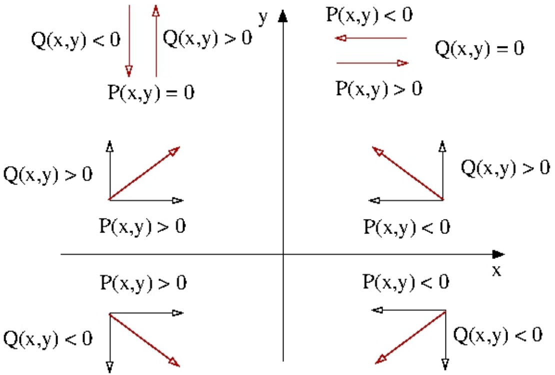
$$\begin{cases} \dfrac{dx}{dt} = P(x,y) \\ \dfrac{dy}{dt} = Q(x,y) \end{cases}$$

- Note that $\dfrac{Q(x,y)}{P(x,y)} = \dfrac{dy}{dx} = \tan\alpha$, where α is an angle between X-axis and the vector, tangent to the solution (trajectory).



Vector field

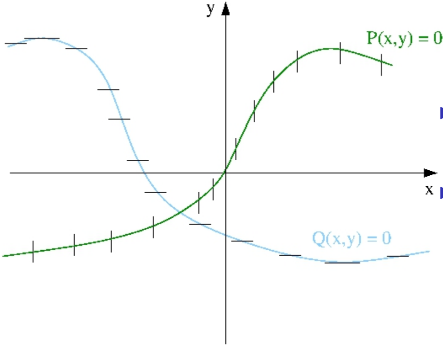
$$\begin{cases} \dfrac{dx}{dt} = P(x, y) \\ \dfrac{dy}{dt} = Q(x, y) \end{cases}$$



- Sign of the functions $P(x, y)$ and $Q(x, y)$ determine the direction of the vector field.
- Their absolute values — the vector magnitudes or "speed" of the representative point.

Nullclines

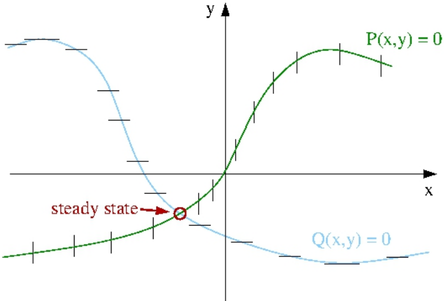
- Note:
  1. $P(x,y) = 0$ given $Q(x,y) \neq 0$
     $\Rightarrow \tan\alpha = \infty \Rightarrow \alpha = \pm\pi/2(90^\circ)$
  2. $Q(x,y) = 0$ given $P(x,y) \neq 0 \Rightarrow \tan\alpha = 0 \Rightarrow \alpha = 0$
- Thus, we know exactly at which angle the trajectories will pass through the lines where $P(x,y) = 0$ or $Q(x,y) = 0$.



- The $P(x,y) = 0$ is called *x-nullcline*.
- The $Q(x,y) = 0$ is called *y-nullcline*.

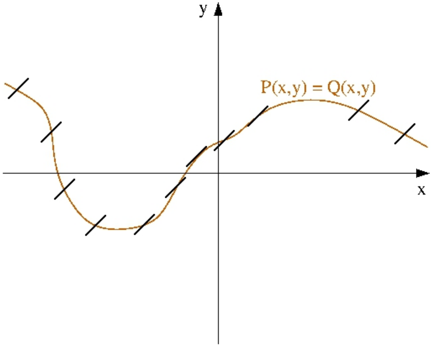
Nullclines

- When two conditions are fulfilled simultaneously, i.e. $P(x,y) = 0$ and $Q(x,y) = 0$, the derivatives $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} = 0$.
- That is, the system does not move in time, hence, the steady state.
- Thus, the intersections of the two curves $P(x,y) = 0$ and $Q(x,y) = 0$ on the phase plane are *steady states* of the system.



Other nullclines

- One can have other nullclines crossing the trajectories at other angles α.
- If $\dfrac{Q(x,y)}{P(x,y)} = A$, then $\alpha = \arctan A$.
- For example, if $A = 1$, then $\alpha = 45^\circ$.
- These sometimes called *minor nullclines* (as opposed to the *major* ones with $\alpha = 0^\circ$ and 90°).



Sketching the vector field

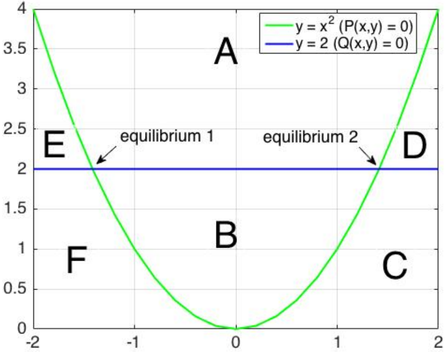
- The major nullclines divide the phase space into regions.
- In each region one must determine the signs of the functions $P(x, y)$ and $Q(x, y)$. For this it is enough to "sample" one point from the region and plug into the functions. The whole region will have the *similarly* oriented vector field.
- At the major nullclines (i.e. crossing the border of the region) either $P(x, y) = 0$ or $Q(x, y) = 0$ and the vector field changes directions.
- Here by the direction of the vector field in the regions we mean four main directions considered above: north-east, south-east, south-west, north-west.

Sketching the vector field: example

- Consider the system:

$$\begin{cases} \dfrac{dx}{dt} = y - x^2 \\ \dfrac{dy}{dt} = y - 2 \end{cases}$$

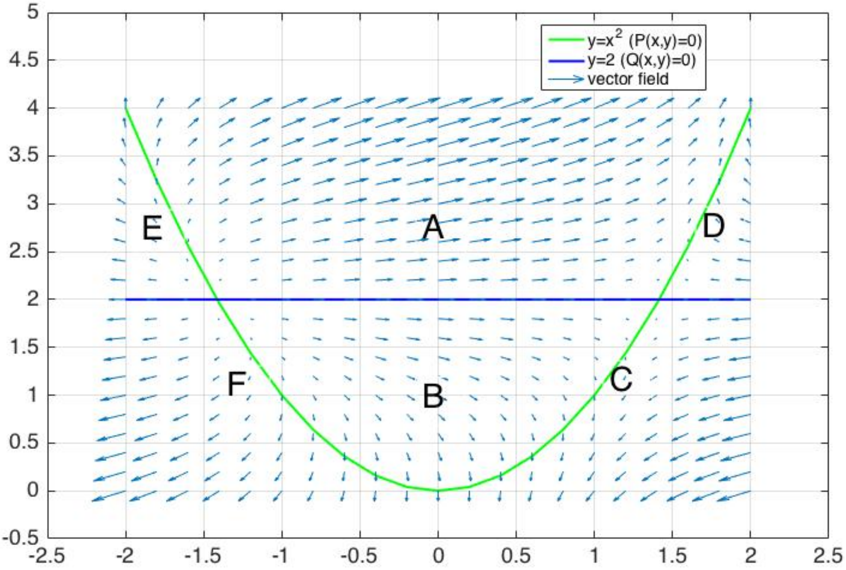
- $P(x, y) = 0 \Rightarrow y = x^2$ and $Q(x, y) = 0 \Rightarrow y = 2$.



Sketching the vector field: example

- Now we can probe all six regions to determine signs of $P(x, y)$ and $Q(x, y)$.
- A: $P(0, 3) = 3 > 0$, $Q(0, 3) = 1 > 0$ (northeast).
- B: $P(0, 1) = 1 > 0$, $Q(0, 1) = -1 < 0$ (southeast).
- C: $P(2, 1) = -3 < 0$, $Q(2, 1) = -1 < 0$ (southwest).
- D: $P(2, 3) = -1 < 0$, $Q(2, 3) = 1 > 0$ (northwest).
- E: $P(-2, 3) = -1 < 0$, $Q(-2, 3) = 1 > 0$ (northwest).
- F: $P(-2, 1) = -3 < 0$, $Q(-2, 1) = -1 < 0$ (southwest)

Sketching the vector field: example



Sketching the vector field: example

- Based on the vector field we can even see the *stability* of the two equilibrium points.
- All flows tend away from the left equilibrium, thus it is unstable.
- However, in some subregions of the phase space the vector field is pointed toward the right equilibrium, whereas the other subregions contain flows directed away from it. Thus, the right equilibrium is a saddle.

Stability analysis

- There are various notions on stability according to: Lyapunov, Poisson etc.
- We encountered the Lyapunov stability in the lecture on linear planar systems.
- We will use those results here as well.
- The idea behind the stability analysis is to unerstand the behavior of the **linearized** system in a close vicinity of the steady states.
- How to linearize a generally nonlinear system?
- First, we apply small perturbations to the system at a steady state.
- Then, we look at how the system behaves in time starting from the perturbed state (we could equally look at how the *perturbation* behaves in time, but the algebraic expressions below would be slightly different).

Linear stability analysis

- The steady states of the system:

$$\begin{cases} P(x,y) = 0 \\ Q(x,y) = 0 \end{cases} \Rightarrow \begin{cases} x = \bar{x} \\ y = \bar{y} \end{cases}$$

- Apply small perturbation (ξ, η) at the equilibrium $(\bar{x}, \bar{y})$:

$$x = \bar{x} + \xi, \qquad y = \bar{y} + \eta$$

- Plug new perturbed (x, y) into the equations:

$$\begin{cases} \dfrac{d(\bar{x}+\xi)}{dt} = P(\bar{x}+\xi, \bar{y}+\eta) \\ \dfrac{d(\bar{y}+\eta)}{dt} = Q(\bar{x}+\xi, \bar{y}+\eta) \end{cases}$$

Linear stability analysis

$$\begin{cases} \dfrac{d\bar{x}}{dt} + \dfrac{d\xi}{dt} = P(\bar{x} + \xi, \bar{y} + \eta) \\ \dfrac{d\bar{y}}{dt} + \dfrac{d\eta}{dt} = Q(\bar{x} + \xi, \bar{y} + \eta) \end{cases}$$

- $\dfrac{d\bar{x}}{dt} = \dfrac{d\bar{y}}{dt} = 0$ by the definition of equilibrium.
- Functions $P(\bar{x} + \xi, \bar{y} + \eta)$ and $Q(\bar{x} + \xi, \bar{y} + \eta)$ can be expanded into the Taylor series about the equilibrium point $(\bar{x}, \bar{y})$.
- While expanding the functions, we can take only the first order terms (hence, the *linear* analysis).
- Taylor series of a function $f(z)$ around a point $z = a$:

$$f(z) = f(a) + f'(a)(z - a) + \dots$$

Linear stability analysis

- Similarly, the Taylor series of a function in two variables $f(u,v)$ around point (u_0,v_0):

$$f(u,v)=f(u_0,v_0)+f'_u(u_0,v_0)\cdot(u-u_0)+f'_v(u_0,v_0)\cdot(v-v_0)+\dots$$

- So, given $(u_0,v_0)=(\bar{x},\bar{y})$, $u-u_0=x-\bar{x}\equiv\xi$ and $v-v_0=y-\bar{y}\equiv\eta$:

$$P(x,y)=P(\bar{x},\bar{y})+P'_x(\bar{x},\bar{y})\cdot\xi+P'_y(\bar{x},\bar{y})\cdot\eta+\dots$$
$$Q(x,y)=Q(\bar{x},\bar{y})+Q'_x(\bar{x},\bar{y})\cdot\xi+Q'_y(\bar{x},\bar{y})\cdot\eta+\dots$$

- Additionally, remembering that at equilibrium $P(\bar{x},\bar{y})=Q(\bar{x},\bar{y})=\frac{d\bar{x}}{dt}=\frac{d\bar{y}}{dt}=0$ and $x=\bar{x}+\xi$, $y=\bar{y}+\eta$:

$$\begin{cases}\frac{d\xi}{dt}=P'_x(\bar{x},\bar{y})\cdot\xi+P'_y(\bar{x},\bar{y})\cdot\eta\\ \frac{d\eta}{dt}=Q'_x(\bar{x},\bar{y})\cdot\xi+Q'_y(\bar{x},\bar{y})\cdot\eta\end{cases}$$

Linear stability analysis

$$\begin{cases} \dfrac{d\xi}{dt} = P'_x(\bar{x}, \bar{y}) \cdot \xi + P'_y(\bar{x}, \bar{y}) \cdot \eta \\ \dfrac{d\eta}{dt} = Q'_x(\bar{x}, \bar{y}) \cdot \xi + Q'_y(\bar{x}, \bar{y}) \cdot \eta \end{cases}$$

- Note that we assumed equality above, although it is an approximation by the linear terms of the Taylor series.
- Clearly, this form now reminds us about the planar linear systems we had before:

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} P'_x(\bar{x}, \bar{y}) & P'_y(\bar{x}, \bar{y}) \\ Q'_x(\bar{x}, \bar{y}) & Q'_y(\bar{x}, \bar{y}) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

- Confer with what we had in the linear planar systems:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Linear stability analysis

- $X' = AX$, where

$$A = \begin{pmatrix} P'_x(\bar{x}, \bar{y}) & P'_y(\bar{x}, \bar{y}) \\ Q'_x(\bar{x}, \bar{y}) & Q'_y(\bar{x}, \bar{y}) \end{pmatrix}$$

- Matrix A is called *Jacobian*.
- Jacobian can be designated as J, i.e. $A = J$.
- Stability of an equilibrium is determined by the regular procedure of finding eigenvalues of the Jacobian.

Example

- Let us consider the example we had earlier in this lecture:

$$\begin{cases} \frac{dx}{dt} = y - x^2 \\ \frac{dy}{dt} = y - 2 \end{cases}$$

- $P(x, y) = y - x^2$, $Q(x, y) = y - 2$
- Steady states:

$$\begin{cases} y - x^2 = 0 \\ y - 2 = 0 \end{cases} \Rightarrow \begin{cases} y = 2 \\ x = \pm\sqrt{2} \end{cases}$$

- Thus, $(\bar{x}_1, \bar{y}_1) = (-\sqrt{2}, 2)$ and $(\bar{x}_2, \bar{y}_2) = (\sqrt{2}, 2)$.

Example (continued)

- $P'_x = -2 \cdot x$, $P'_y = 1$, $Q'_x = 0$, $Q'_y = 1$
- $(\bar{x}_1, \bar{y}_1) \Rightarrow J = \begin{pmatrix} 2\sqrt{2} & 1 \\ 0 & 1 \end{pmatrix}$
- $(\bar{x}_2, \bar{y}_2) \Rightarrow J = \begin{pmatrix} -2\sqrt{2} & 1 \\ 0 & 1 \end{pmatrix}$
- $(\bar{x}_1, \bar{y}_1)$:

$$\begin{vmatrix} 2\sqrt{2} - \lambda & 1 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - (2\sqrt{2} + 1)\lambda + 2\sqrt{2} = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{2\sqrt{2} + 1 \pm \sqrt{(2\sqrt{2} + 1)^2 - 4 \cdot 2\sqrt{2}}}{2}$$

$$\Rightarrow \lambda_{1,2} = \frac{2\sqrt{2} + 1 \pm (2\sqrt{2} - 1)}{2} = 1 \wedge 2\sqrt{2}$$

- Thus, $(\bar{x}_1, \bar{y}_1)$ is an **unstable node**.

Example (continued)

- Similarly for the second equilibrium $(\bar{x}_2, \bar{y}_2)$:

$$\begin{vmatrix} -2\sqrt{2} - \lambda & 1 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - (1 - 2\sqrt{2})\lambda - 2\sqrt{2} = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{1 - 2\sqrt{2} \pm \sqrt{(1 - 2\sqrt{2})^2 + 8\sqrt{2}}}{2}$$

$$\Rightarrow \lambda_{1,2} = \frac{1 - 2\sqrt{2} \pm (1 + 2\sqrt{2})}{2} = -2\sqrt{2} \wedge 1$$

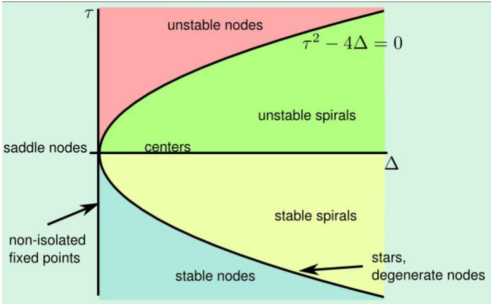
- Thus, $(\bar{x}_2, \bar{y}_2)$ is a **saddle**.
- *Compare these results with the qualitative analysis of the vector field we made earlier in this lecture.*

Back to nonlinear system

- In the previous example we linearized the nonlinear system and made direct association of the node and saddle type equilibria of the linearized system with the original nonlinear system.
- A nonlinear system around (in small enough vicinity of) equilibria can be roughly approximated with the linearized equations and the vector field around equilibria will be conjugate to that of the linearized system around the equilibria.
- But this is *only for hyperbolic* systems, i.e. those without zero eigenvalues λ_i.
- If any $\lambda_i = 0$ then in order to be able to determine stability one needs to consider higher order polynomials in Taylor series. Moreover, small variations in the nonlinear functions $P(x, y)$ and $Q(x, y)$ can change properties of equilibria and, thus, lead to *bifurcations*.

Non-hyperbolic systems

- Recall that $\lambda_{1,2} = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$, where τ and Δ — trace and determinant of the linearized matrix of coefficients, respectively.



- Non-hyperbolic situation can occur:
  - on the line $\Delta = 0$: redundant case, non-isolated equilibria located on a line (the whole line is equilibria).
  - on a part of the line $\tau = 0$, where $\Delta > 0$: center points.

Picture from Wikipedia: http://en.wikipedia.org/wiki/Linear_dynamical_system

$\Delta = 0$ (redundant case)

- Consider the system:

$$\begin{cases} x' = x^2 \\ y' = -y \end{cases}$$

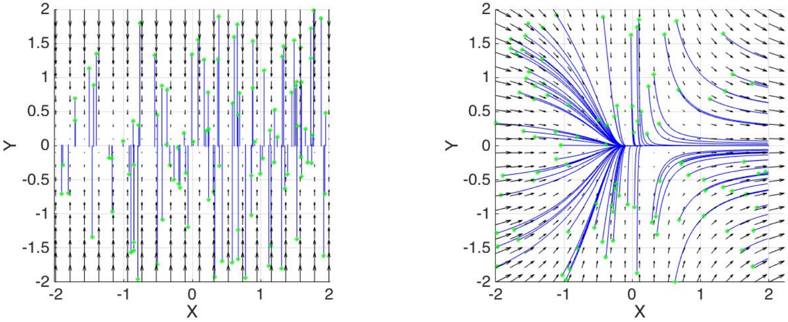


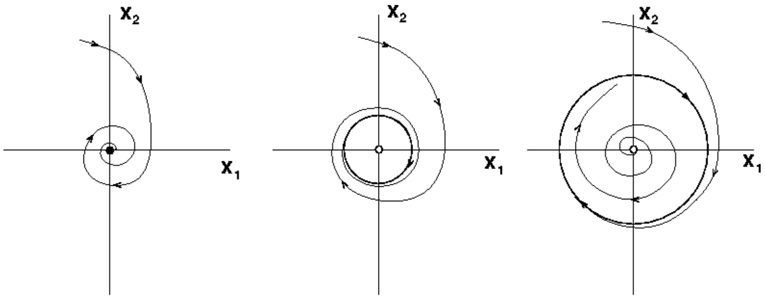
Figure: Linear (left) and nonlinear (right) phase portraits differ significantly.

Center

- For $\tau = 0$ and $\Delta > 0$, λ_i are purely imaginary.
- When $\operatorname{Re} \lambda_i = 0$ and $\lambda_{1,2} = \pm 2\sqrt{\Delta} i$, linear system has a center point, but in nonlinear systems with imaginary eigenvalues $\operatorname{Re} \lambda_i = 0$ is a condition for Andronov-Hopf bifurcation (or simply Hopf).
- Hopf bifurcation is the birth of a *limit cycle* from an equilibrium.
- Limit cycle is a closed orbit in the phase space corresponding to the repeated motion, oscillations, that is true nonlinear oscillations with dissipation.

Supercritical Hopf bifurcation

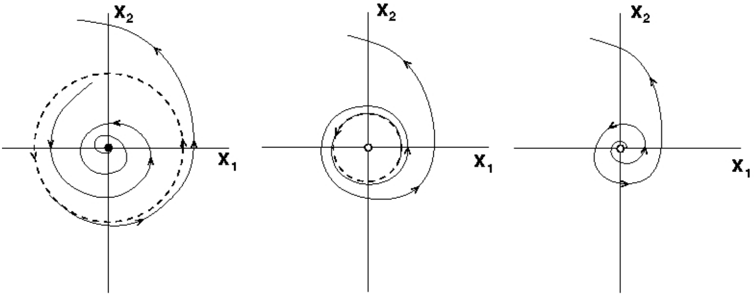
- Before the critical value of the parameter ($\alpha < 0$) stable focus exists; at the critical value ($\alpha = 0$) the limit cycle emerges whereas the focus loses stability; after it ($\alpha > 0$) exists the limit cycle and unstable equilibrium $\Rightarrow$ all trajectories tend toward the limit cycle.



$\alpha < 0$ $\alpha = 0$ $\alpha > 0$

Subcritical Hopf bifurcation

- Before the critical value of the parameter ($\alpha < 0$) stable equilibrium and unstable limit cycle exist; at the bifurcation value ($\alpha = 0$) the limit cycle merges with the equilibrium and the latter loses stability $\Rightarrow$ unstable equilibrium ($\alpha > 0$).



$\alpha < 0$ $\alpha = 0$ $\alpha > 0$

Summary

- We have considered simple geometrical techniques (nullclines, vector fields) for analysis of the global dynamics of nonlinear systems.
- We have also seen a way to understand stability of the nonlinear dynamical system around its equilibria, using the linearized form of the equations.
- Additionally, we have discussed the non-hyperbolic systems where the linear stability analysis does not give an answer. Moreover, these systems demonstrate new kind of bifurcations (Hopf bifurcation).
- To get deeper into this read Hirsch et al. book, Chapter 8–9.