Dynamical Systems and Chaos
Part I: Theoretical Techniques

Lecture 4: Discrete systems + Chaos

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Discrete maps

$$x_{n+1} = f(x_n)$$

- Discrete time steps.
- x_0 is a seed of a sequence of numbers x_n produced by the map:

$$x_0, x_1 = f(x_0), x_2 = f(f(x_0)), x_3 = f(f(f(x_0))), \ldots$$

- We denote the n-fold composition of the function f with itself as f^n.
- Thus $x_0, x_1 = f(x_0), x_2 = f^2(x_0), x_3 = f^3(x_0), \ldots$.
- Example: $f(x) = 2x + 1$, given $x_0 = 0$. Then the sequence is:

$$0, 1, 3, 7, 15, 31, 63, \ldots$$

- **Advantage:** direct iteration of the map instead of finding approximate or exact analytical solutions.

Steady states and periodic orbits

- If x_0 is a steady state, then $f(x_0) = x_0$.
- Periodic orbit of *period-n* is an orbit where n is the least positive integer for which $f^n(x_0) = x_0$. Such orbits are also called *n-cycles*.
- Example: $f(x) = 2x - 1$. $x_0 = 1$ is a steady state, since $f(1) = 2 \cdot 1 - 1 = 1$.
- Example: $f(x) = -x^3$. $x_0 = \pm 1$ produce period-2 orbits, since $f(1) = -1, f(-1) = 1$ and $f(-1) = 1, f(1) = -1$.

Visualization of maps

- One useful way to visualize the dynamics of maps is to use so called *Lamerey diagrams*.
- One is to draw $y = f(x)$ and the diagonal $y = x$ on the same graph.
- Start with the point (x_0, x_0) on the diagonal.
- Draw a vertical line up to the graph $y = f(x)$ (this will be point (x_0, x_1)).
- Draw a horizontal line to the diagonal (point (x_1, x_1)).
- Continue iteration.

Graphical iteration

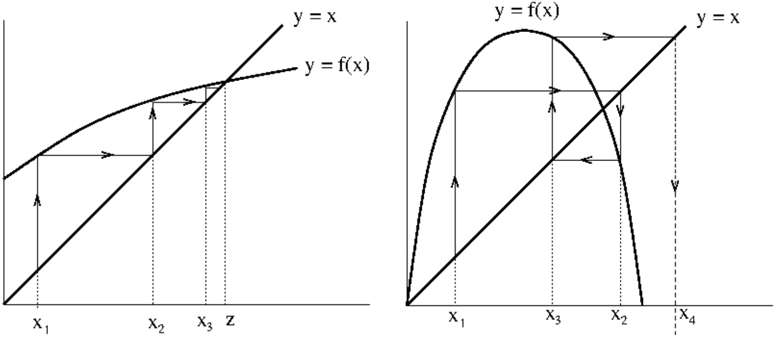


Figure: Examples of the Lamerey diagrams. Trajectory converges to z at the left panel, while it expands (at least for the time steps shown) on the right.

Stability of equilibria

- Steady states in a discrete dynamical systems can be *stable* (sink), *unstable* (source), or *neutral*.
- If x_0 is a sink-equilibrium (*stable*), then there exists a neighborhood $\mathbb{U}$ in the vicinity of x_0 such that if $y_0 \in \mathbb{U}$, then $f^n(y_0) \in \mathbb{U}$ for all n and $f^n(y_0) \to x_0$ as $n \to \infty$.
- In other words, one can find a region around the equilibrium x_0, where all points will map to the same region and the mapped values will tend (with number of iterated steps) to the equilibrium x_0.
- For the source-equilibrium x_0 (*unstable*) all points from within the region around x_0 will tend outside the region with increasing number of iterations.
- There are also neutral steady states. This takes place if neither of the above is true.

Stability

- Analytically stability of the equilibria can be assessed with:
  - x_0 is a sink, if $|f'(x_0)| < 1$.
  - x_0 is a source, if $|f'(x_0) > 1$.
  - if $f'(x_0) = \pm 1$, nothing can be said about the stability of x_0.

Stability of periodic orbits

- Note that period n orbits are fixed points of the operator f^n. In other words, $f^n(x_0) = x_0$.
- Thus, one can classify the periodic orbits as sinks or sources depending on whether $|(f^n)'(x_0)| < 1$ or $|(f^n)'(x_0)| > 1$, respectively.

Logistic map

$$f(x) = \lambda x(1 - x)$$

- Steady states: $\lambda x(1 - x) = x$, so $\bar{x}_1 = 0$ and $\bar{x}_2 = \dfrac{\lambda - 1}{\lambda}$.
- Stability: $f'(x) = \lambda - 2\lambda x$, so $f'(0) = \lambda$, $f'(\frac{\lambda-1}{\lambda}) = 2 - \lambda$.
- Thus, $\bar{x}_1$ is stable if $\lambda \in (-1, 1)$; $\bar{x}_2$ is stable if $1 < \lambda < 3$.
- $\bar{x}_1$ and $\bar{x}_2$ cannot be stable simultaneously.

Logistic map

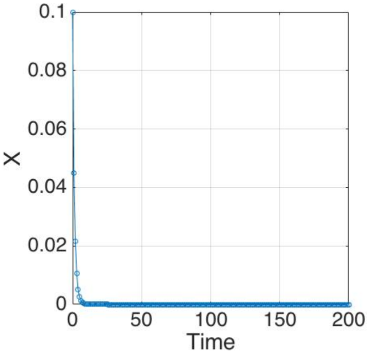


Figure: Time series (left) and graphical iteration (right) of the logistic map $f(x) = \lambda x(1 - x)$. $\lambda = 0.5$, $\bar{x}_1 = 0$ is stable.

Logistic map

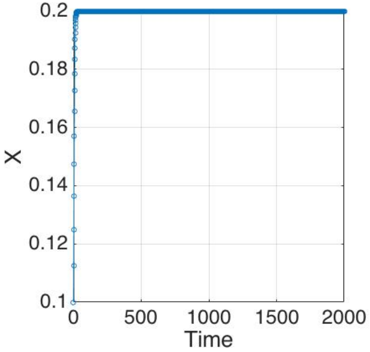


Figure: Time series (left) and graphical iteration (right) of the logistic map $f(x) = \lambda x(1 - x)$. $\lambda = 1.25$, $\bar{x}_1 = 0$ is unstable, whereas $\bar{x}_2 = \frac{\lambda - 1}{\lambda} = 0.2$ is stable.

Logistic map

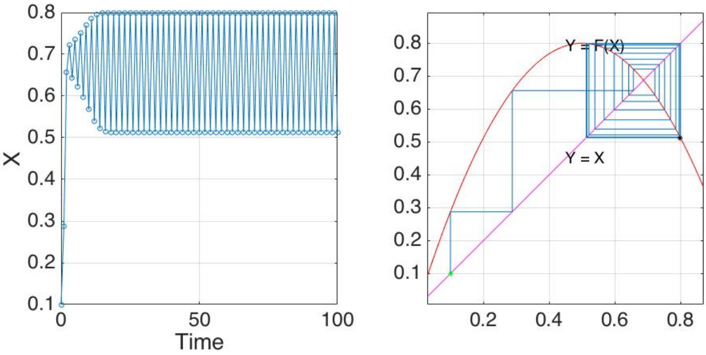


Figure: Time series (left) and graphical iteration (right) of the logistic map $f(x) = \lambda x(1-x)$. $\lambda = 3.2$, both $\bar{x}_1 = 0$ and $\bar{x}_2 = \frac{\lambda - 1}{\lambda} = 0.69$ are unstable. Instead period-2 cycle emerged.

Logistic map: 2-cycle

- To find periodics we need to solve: $f^n(x) = x$. Particularly we need: $f^2(x) = x$. Let us see this graphically.

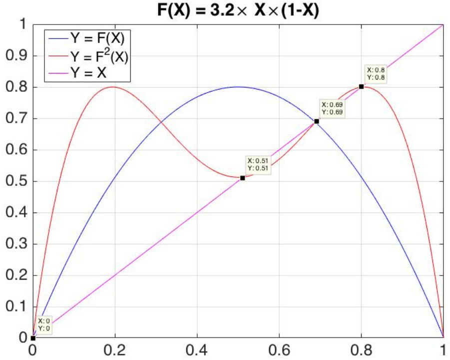
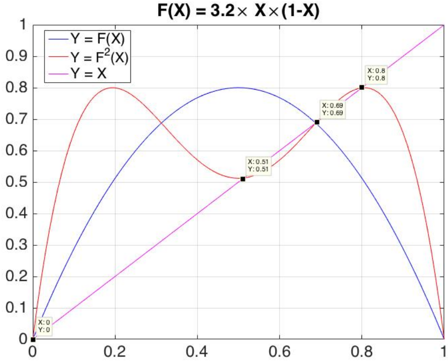


Figure: The graph $f^2(x)$ has four intersection points with $y = x$. Two of them $x = 0$ and $x = 0.69$ are common with $f(x) = x$ and not periodics, but the steady states. Two other points $x = 0.51$ and $x = 0.8$ are the cycle-2 points.

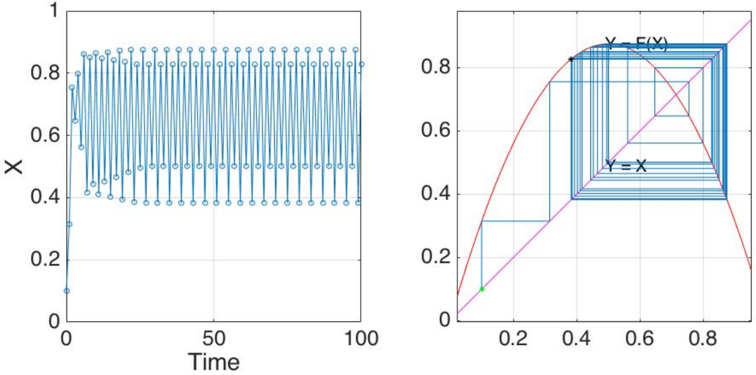
Logistic map: 2-cycle

- Note from the figure, that the derivative $|(f^2)'(0.51)| < 1$ and $|(f^2)'(0.8)| < 1$. Thus, it is a stable 2-cycle.
- Also note that when $f'(x_0) < 0$ for a steady state x_0, the trajectory jumps around the steady state, be it stable or unstable. That is, the trajectory does not approach or leave the steady state from one side. $f'(0.69) < 0$ makes a periodic orbit jump around the steady state, hence, cycle.



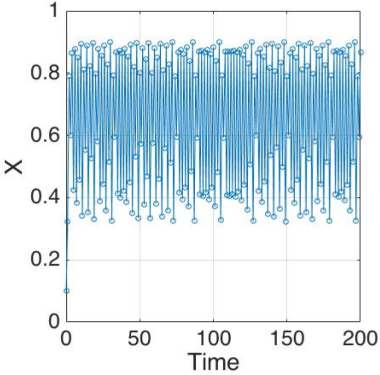
Logistic map: 4-cycle

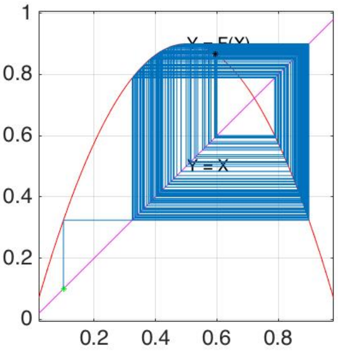
- $\lambda = 3.5$



Logistic map: chaos?

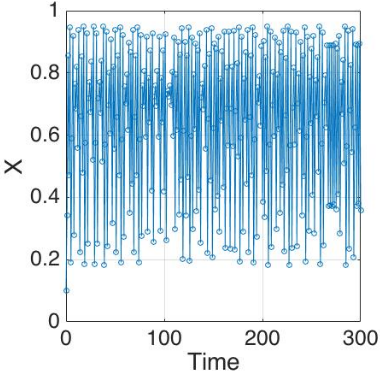
- $\lambda = 3.6$

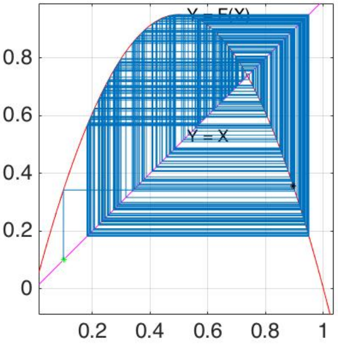




Logistic map: chaos?

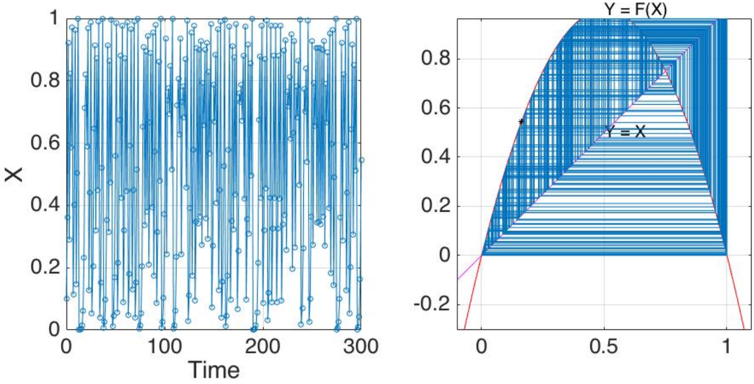
- $\lambda = 3.8$



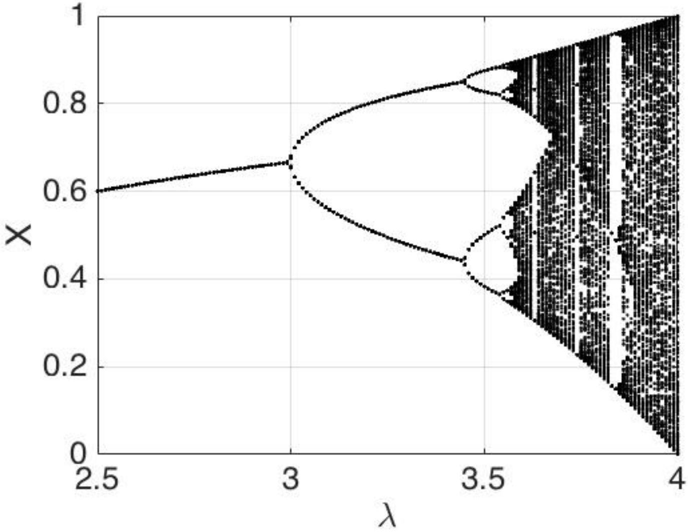


Logistic map: chaos

- $\lambda = 4.0$
- The entire region $(0, 1)$ is covered with points.



Logistic map: diagram



Chaos

- *Deterministic chaos* is a phenomenon when small discrepancies in the initial conditions lead to unpredictable behaviour.
- Importantly: the system stays **deterministic** (no random forces exist), although the outcome is somewhat unpredictable.
- There is no universally accepted definition for chaos, but the following three conditions must be satisfied:
  1. Sensitivity to the initial conditions.
  2. Topological mixing: over time any given region overlaps with any other given region of the phase space.
  3. Dense periodic orbits: every point in the space is approached arbitrarily closely by periodic orbits.
- Chaos can be observed in ODE systems, but the minimal dimension must be 3.

Chaos: a brief history

- Chaotic regimes were shown in the last half of the 20th century, although in the turn of the century Henri Poincaré in his 1903 essay "Science and Method" wrote:

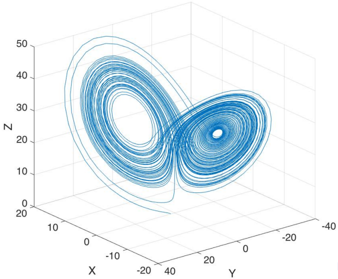
  If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that *small differences in the initial conditions produce very great ones* in the final phenomena. A **small error** in the former will produce an **enormous error** in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon. (the emphasis is mine).

- Edward Lorenz observed chaos in a simple 3D system (1963).

Lorenz attractor

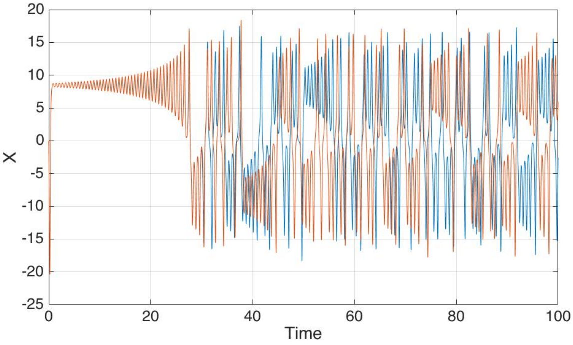
$$\begin{cases} x' = \sigma y - \sigma x \\ y' = rx - y - xz \\ z' = xy - bz \end{cases}$$

- Note that the system has only two nonlinear terms.
- Originally the system was intended to model a meteorological phenomenon.
- Lorenz observed a chaotic attractor for $\sigma = 10$, $r = 28$, $b = 8/3$:



Lorenz attractor: sensitivity

- The trajectories seem to rotate around the same two centers.
- However, any two very closely situated initial points eventually diverge in time significantly.
- Consider two trajectories originating from two slightly different initial conditions: $P_1 = (0, -10.0, 0)$ and $P_2(0, -10.005, 0)$:



Chaotic attractor

Lorenz attractor is chaotic *not* because of:

- external random forces (they do not exist in the system)
- infinite number of variables (there are only 3)
- quantum indeterminacy (the system is classical)

Deterministic chaos appears in systems with:

- nonlinear interactions
- instabilities

Stability

- Stability can be determined not only for equilibria, but for the trajectories in the phase space.
- Stability can be defined differently: according to Lyapunov, asymptotic stability, according to Poisson.
- Instead of applying small perturbation to the steady state, we can apply it to any particular solution $x^0(t)$ of $x' = F(x)$. Thus, we have the perturbation $y(t) = x(t) - x^0(t)$ and $y' = F(x^0 + y) - F(x^0)$.
- Note, now we **need** to look strictly at the time evolution of the perturbation, not that of the system in a perturbed state (as we did in the linear analysis of equilibria we had in Lecture 3, there we could do both).
- By linearizing we get:

$$y' = J(t)y\,,$$

J is a Jacobian matrix.

Stability: eigenvalues

- The eigenvalues ρ_i of Jacobian J can be found from:

  $$|J - \rho I| = 0\,,$$

  I is again an identity matrix.
- Thus, the initial perturbation at time $t = t_0$ changes over time along the eigenvector V_i as follows:

  $$y^i(t) = y^i(t_0)\mathrm{e}^{(t-t_0)\rho_i}$$

- $\mathrm{Re}\,\rho_i$ determines increase ($\mathrm{Re}\,\rho_i > 0$) or decrease ($\mathrm{Re}\,\rho_i < 0$) in amplitude of the corresponding perturbation.
- Note that generally Jacobian is a time-dependent matrix $J(t)$, hence the eigenvalues $\rho_i(t)$ and eigenvectors $V_i(t)$ are time-dependent entities too.
- Therefore, $y^i(t)$ may be increasing and decreasing at different points of the given trajectory $x^0(t)$.

Lyapunov exponent

- Stability of the small perturbation along the eigenvector V_i is determined by the *Lyapunov characteristic exponent*:

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t - t_0} \ln \frac{|y^i(t)|}{|y^i(t_0)|}$$

- N-dimensional system has N Lyapunov exponents for a given trajectory $x^0(t)$, which being arranged from the biggest to the smallest, form the *spectrum of Lyapunov exponents*:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$$

Lyapunov exponent: connection with eigenvalues

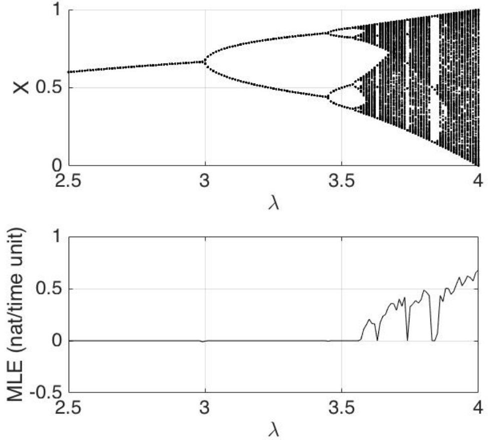
- It can be shown that

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t - t_0} \int_{t_0}^{t} \operatorname{Re} \rho_i(t')dt'$$

- That is, λ_i is an averaged along the given trajectory $x^0(t)$ real part of ρ_i.
- Thus, λ_i shows how the initial perturbation changes on average along the trajectory.
- Trajectory $x^0(t)$ is stable if on average the initial perturbation $y(t_0)$ does not grow along the given trajectory. For this to be true, the spectrum of Lyapunov exponents *must not* contain positive values.
- Recall chaos is connected with *instabilities*. Thus, at least one positive λ_i is a characteristic sign of chaos.

Maximal Lyapunov exponent (MLE)

- Practically, one calculates Maximal Lyapunov Exponent (MLE, λ_1 in the spectrum). If that is positive, then there is a chaotic attractor.
- Recall the diagram for the logistic map $f(x) = \lambda x(1 - x)$.



Stability of chaotic behaviour

- Regular attractors (steady states, limit cycles) are fully stable according to Lyapunov or fully unstable.
- This is not so for chaotic attractors. Chaotic trajectory is unstable at least in one direction, i.e. at least one Lyapunov exponent is positive (when there are more than one positive Lyapunov exponent in the spectrum it is said *hyper-chaos*).
- Instability and attracting character of the chaotic attractor do not contradict.
- The initial points tend toward the attractor, but on the attractor diverge from each other (recall transitivity property of chaos).
- There is another type of stability — Poisson stability. Stable according to Poisson indicates the trajectory does not leave a confined region of the phase space staying there for arbitrarily long.
- Chaotic trajectory is stable by Poisson, but unstable by Lyapunov.

Dimension of chaotic attractor

- Strange attractor is characterized by complex geometrical structure with non-integer dimension.
- The dimension can be assessed via:

$$D_L = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|},$$

where j is maximal integer for which $\lambda_1 + \lambda_2 + \ldots + \lambda_j \geq 0$.

- D_L is called Lyapunov dimension.
- $D_L = 0$ for equilibria.
- $D_L = 1$ for limit cycles (oscillations).
- $D_L = n$ for n-dimensional tori.
- Regular attractors (equilibria, limit cycles, tori) have D_L equal to the metric dimension and stable by Lyapunov trajectories on the attractors.
- Chaotic/strange attractor is characterized with non-integer (Lyapunov) dimension.

Chaos and strange-ness of attractor

- Usually, *chaotic* and *strange*, when referring to an attractor, are interchangeably used, but...
- Not all chaotic attractors are strange (by strange we mean complex geometrical structure with non-integer dimension, e.g. Lyapunov dimension).
- There are strange attractors with trajectories stable by Lyapunov (all λ_i are non-positive).
- On the other hand, there are regular (non-strange) attractors with diverging trajectories on them (some λ_i are positive).

Summary

- Maps are simpler to simulate and analyze than ODE-based dynamical systems.
- Maps are capable of producing complex behaviour, like chaos, for lower dimensions than systems of ODE's (logistic map is 1D, Lorenz is 3D).
- Chaotic behaviour is characterized by *deterministic* law of evolution and *unpredictability*. The two seem to be "incompatible".
- Lyapunov exponents are the first things to check when searching for chaos.
- Revise the material in Hirsch et al. book, Chapters 14.1–14.3, 14.5, 15.1–15.4.