Dynamical Systems and Chaos
Part I: Theoretical Techniques

Lecture 5: Hamiltonian vs. dissipative systems

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Conservative and dissipative

- Any system can be classified as *conservative* or *dissipative*.
- Conservative systems have constant entities (usually, energy). Physically we mean systems with no *influx* and no *production* of energy/matter (Note that it is possible to construct a system with a constant entity, but with influx of energy and/or matter).
- Dissipative systems lose energy with time. In order to maintain persistent behaviours the dissipative system must have influx of energy/matter.
- There are many systems (especially, from classical physics) that are conservative.
- Biological systems are always dissipative, thus we focus on the dissipative systems in this course.
- But we need to stress important differences.

Hamiltonian systems

- Hamiltonian systems are the special type of dynamical systems arising in classical mechanics.
- We consider planar Hamiltonian systems of the form:

$$\begin{cases} x' = \dfrac{\partial H}{\partial y}(x, y) \\ y' = -\dfrac{\partial H}{\partial x}(x, y) \end{cases}$$

  where $H(x, y)$ is a *Hamiltonian function.*
- Hamiltonian function is a *constant of motion*, that is $H(x, y)$ is constant along every solution of the system: $\dot{H} \equiv 0$ as:

$$\dot{H} = \frac{\partial H}{\partial x}x' + \frac{\partial H}{\partial y}y' = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial H}{\partial y}\left(-\frac{\partial H}{\partial x}\right) = 0$$

Some examples

- Harmonic oscillator

$$\begin{cases} x' = y \\ y' = -kx \end{cases}$$

- The Hamiltonian function of the system above is: $H(x, y) = \frac{1}{2}y^2 + \frac{k}{2}x^2$.
- Ideal pendulum

$$\begin{cases} \theta' = \nu \\ \nu' = -\sin\theta \end{cases}$$

- The total energy is a Hamiltonian of the system: $H(\theta, \nu) = \frac{1}{2}\nu^2 + 1 - \cos\theta$.

Hamiltonian systems are conservative

- Note that Hamiltonian functions define energy of a mechanical system, hence, the Hamiltonian systems are truly conservative systems.
- Note also that there are infinite number of functions $H(x, y)$, since a constant added to it does not change the condition $\dot{H}(x, y) = 0$.
- Thus, the levels $H(x, y) = Constant$ represent motion with different constant energy.
- If we assume that H is not constant on any open set, we just plot curves $H(x, y) = Constant$ and the solutions of the Hamiltonian system with $H = H(x, y)$ lie on the drawn levels.
- Thus, we need not solve the system analytically or numerically.

Example

- Consider the system:

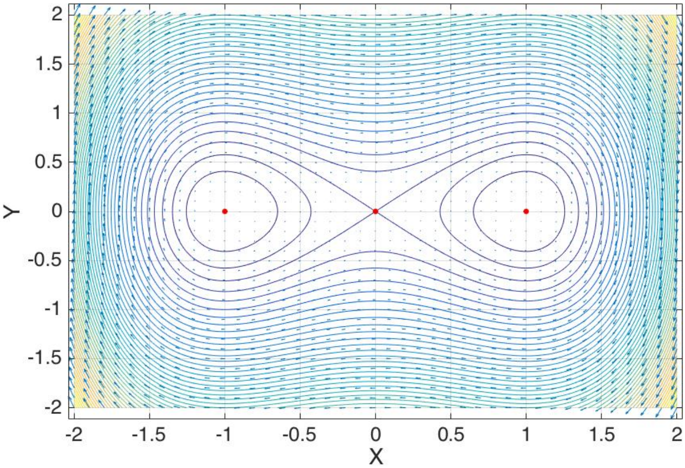
$$\begin{cases} x' = y \\ y' = -x^3 + x \end{cases}$$

  with Hamiltonian $H(x,y) = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}$.

- The linearized system is: $X' = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix} X$
- The steady states of the system are: $(0,0)$, $(\pm 1, 0)$.
- The eigenvalues for the origin $(0,0)$ are ± 1 (saddle), for the $(\pm 1, 0)$ — $\pm\sqrt{2}i$ (center).
- *NOTE: the Hamiltonian systems can only have saddles and centers as their equilibrium points.*

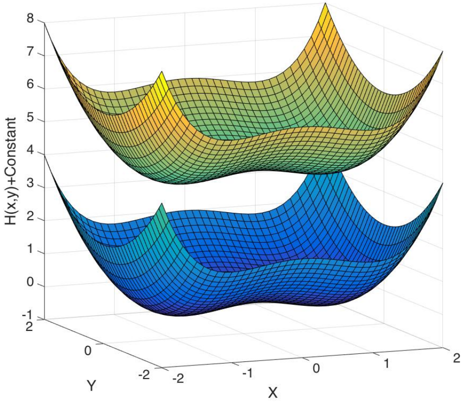
Example (continued)

- Let us plot the levels of the Hamiltonian $H(x, y) = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}$ and the vector field.



Example (continued)

- A constant added to $H(x, y)$ does not change the solutions (vector field), reflecting the fact that the potential energy of a physical system depends on the reference measure.
- Solutions are the lines along the surface with the fixed z-stack $= H(x, y)$.



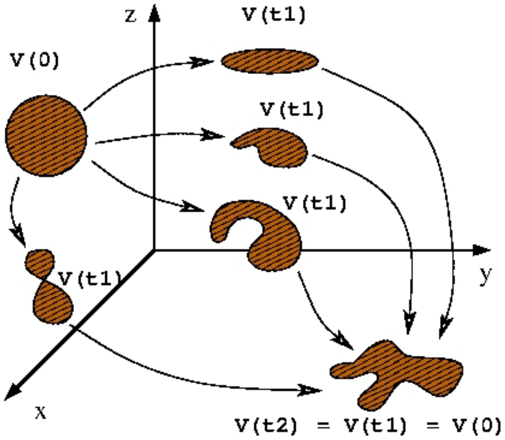
Attractors

- So far we have encountered several types of attractors: equilibrium points, limit cycles, and *strange* attractor (for the chaotic dynamics).
- *Attractor* is an invariant set where points of the phase space tend to either in forward or backward time.
- The sub-region of the phase space starting from which points evolve to eventually reach an attractor, is called *basin of attraction* (obviously, when there is a single attractor, the whole available phase space is its basin of attraction).
- We talk about *volume* of the phase space occupied by the attractor and the phase volume occupied by its basin of attraction.
- For example, an equilibrium point has zero phase volume, but its basin of attraction is presumably a larger set with non-zero phase volume.

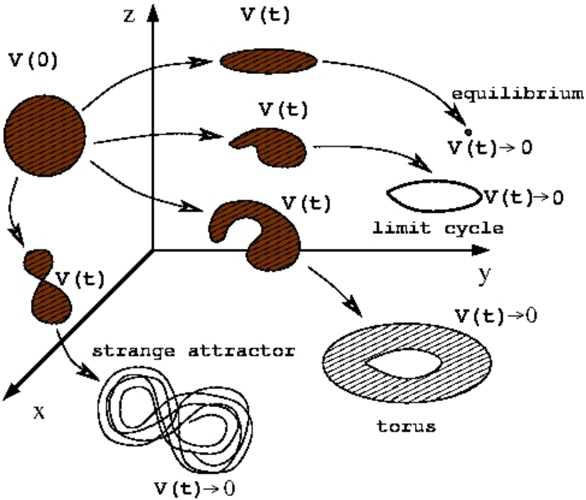
Phase volume

- A set of initial points occupies phase volume $V(0)$. It evolves over time so that at time t volume is $V(t)$.
- For a *Hamiltonian* system $V(t) = V(0)$, that is the volume is **preserved**.
- The initial set can be shrinked in one direction, but this is ultimately accompanied with extension in other directions (Hamiltonian systems).
- Deformations of the initial set can occur, but the phase volume is preserved (Hamiltonian systems).
- For a *dissipative* system $V(t) < V(0)$.

Phase volume in Hamiltonian systems



Phase volume in dissipative systems



Phase volume and attractors

- Attractors assume non-zero volume basins of attraction (although there is no universal definition of an attractor).
- If so, Hamiltonian systems do *not* have attractors, since any initial point is already on the final invariant set (that is the point will be revisited again in time).
- Dissipative systems do have attractors, since they undergo *compression* of the phase volume.
- If we separate attractors from basins of attraction in the definition of attractor, then Hamiltonian systems can also be said to have attractors.
- If a dissipative system starts at its stable equilibrium point, it stays there for arbitrarily long and one cannot see the basin of attraction and compression of the volume. It is already on the attractor.
- Similarly a Hamiltonian system can be understood this way. Any initial point is already on an attractor (and there is inifinite number of them, corresponding to different levels $H(x, y) = Constant$).

Phase volume and Lyapunov exponents

- The average divergence div of the vector field $F(x(t))$ determines the evolution of the phase volume:

$$V(t) = V(t_0)\exp\left[(t - t_0)divF(x(t))\right]$$

- It can be shown, that the sum of Lyapunov exponents can be expressed through the divergence:

$$\sum_{i=1}^{N} \lambda_i = \lim_{t\to\infty} \frac{1}{t - t_0} \int_{t_0}^{t} divF(t')dt'$$

- If the compression of the phase volume takes place, the divergence is negative and:

$$\sum_{i=1}^{N} \lambda_i < 0$$

Example 1

- Ideal harmonic oscillator ($\ddot{x} + \frac{k}{m}x = 0$):

$$\begin{cases} x' = y \\ y' = -\dfrac{k}{m}x \end{cases}$$

- Eigenvalues $\rho_{1,2} = \pm\sqrt{\dfrac{k}{m}}i$. Center.
- The Lyapunov spectrum: $\lambda_1 = \lambda_2 = 0 \Rightarrow \sum \lambda_i = 0$.
- No phase volume compression, no dissipation, no attractor.
- It is a classical Hamiltonian system with

$$H(x, y) = \frac{1}{2}y^2 + \frac{k}{2m}x^2$$

Example 2

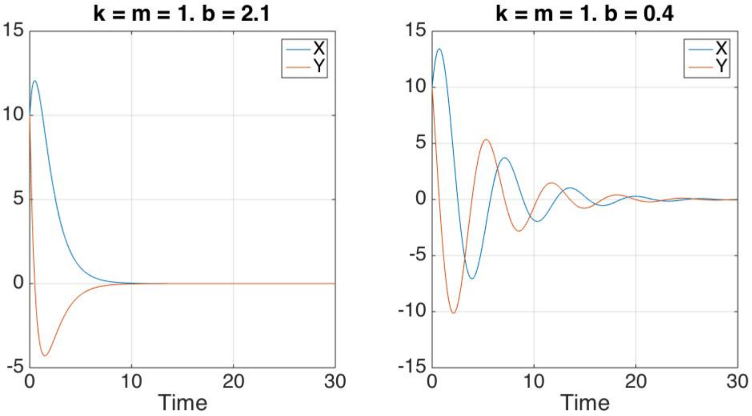
- Harmonic oscillator with friction ($\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$):

$$\begin{cases} x' = y \\ y' = -\dfrac{k}{m}x - \dfrac{b}{m}y \end{cases}$$

- Note b, k, m all positive (physical constants).
- Eigenvalues $\rho_{1,2} = \dfrac{-\frac{b}{m} \pm \sqrt{\left(\frac{b}{m}\right)^2 - 4\frac{k}{m}}}{2}$.
- If $\left(\frac{b}{m}\right)^2 \geq 4\frac{k}{m} \Rightarrow 0 \leq \sqrt{\left(\frac{b}{m}\right)^2 - 4\frac{k}{m}} \leq \frac{b}{m} \Rightarrow \rho_{1,2} < 0$. Stable node.
- If $\left(\frac{b}{m}\right)^2 < 4\frac{k}{m} \Rightarrow \rho_{1,2} = -\frac{b}{2m} \pm \frac{\sqrt{(\frac{b}{m})^2 - 4\frac{k}{m}}}{2} i$. Stable focus (damped oscillations).

Example 2 (continued)

- $k = m = 1$.
- $b = 2.1 \Rightarrow$ stable node.
- $b = 0.4 \Rightarrow$ damped oscillations (stable focus).



Example 2 (continued)

- $k = m = 1$, $b = 2.1$: Lyapunov spectrum: $\lambda_1 = -0.70$, $\lambda_2 = -1.40$, $\sum \lambda_i = -2.10 < 0$, Lyapunov dimension $D_L = 0$.
- $k = m = 1$, $b = 0.4 \Rightarrow$: Lyapunov spectrum: $\lambda_1 = \lambda_2 = -0.20$, $\sum \lambda_i = -0.40 < 0$, Lyapunov dimension $D_L = 0$.
- The system demonstrates phase volume compression, i.e. dissipation ($\sum \lambda_i < 0$).
- Lyapunov dimension $D_L = 0$ (regular attractor, i.e. no complex geometrical structure since $D_L =$ metric dimension).

Example 3

- Brusselator is "chemical" oscillator containing true non-linear oscillations, corresponding to the *limit cycle* attractor.
- Brusselator is modeled using the following equations:

$$\begin{cases} x' = A + x^2 y - (B+1)x \\ y' = Bx - x^2 y \end{cases}$$

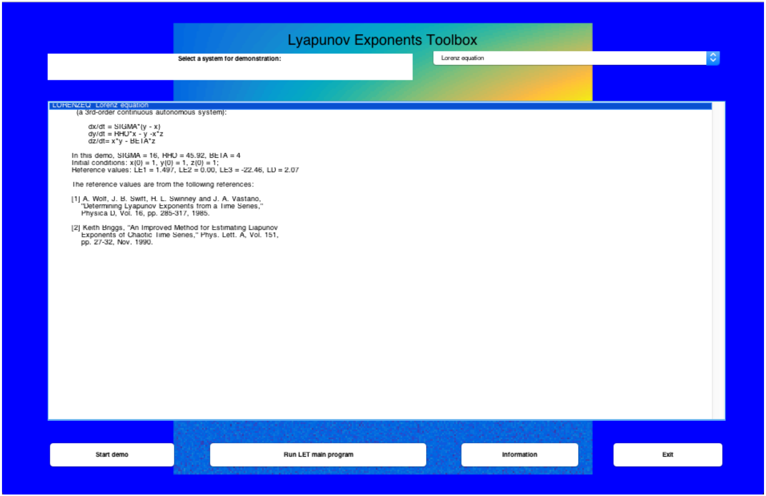
- In the oscillatory regime ($A = 1$, $B = 3$) the Lyapunov spectrum is: $\lambda_1 = -0.001$ (in theory this must be zero), $\lambda_2 = -1.21 \Rightarrow \sum \lambda_i = -1.21$. Lyapunov dimension $D_L = 1 + \frac{0}{1.21} = 1$.
- There is dissipation. Regular attractor since the limit cycle's metric dimension is 1D (line).

Example 4

- Lorenz system:
$$\begin{cases} x' = \sigma(y - x) \\ y' = \rho x - y - xz \\ z' = xy - \beta z \end{cases}$$
where $\sigma = 16$, $\rho = 45.92$, $\beta = 4$
- Lyapunov spectrum: $\lambda_1 = 1.50$, $\lambda_2 = -0.002$ (zero), $\lambda_3 = -22.50 \Rightarrow \sum \lambda_i \approx -21.0$
- Lyapunov dimension $D_L \approx 2.07$.
- There is dissipation with chaos (one λ_i is positive).
- Attractor is strange (non-integer D_L).

Lyapunov Exponents Toolbox (LET)

- LET demo.



Summary

- Hamiltonian systems are important subclass of conservative dynamical systems.
- Biological dynamical systems are dissipative (as opposed to conservative systems), i.e. they have energy/matter losses.
- Dissipation can be assessed by Lyapunov exponents.