Excitability phenomena: Morris-Lecar model*

Consider the following system of equations [1]:

$$
\begin{cases}
C\dfrac{dV}{dt} = -g_L(V - V_L) - g_{Ca}m_\infty(V)(V - V_{Ca}) - g_K w(V - V_K) + I \\[2ex]
\dfrac{dw}{dt} = \lambda_w(v)(w_\infty(V) - w) \\[2ex]
m_\infty(V) = 0.5\left(1 + \tanh\left(\frac{V - V_1}{V_2}\right)\right) \\[2ex]
w_\infty(V) = 0.5\left(1 + \tanh\left(\frac{V - V_3}{V_4}\right)\right) \\[2ex]
\lambda_w(V) = \phi\cosh\left(\frac{V - V_3}{2V_4}\right)
\end{cases}
\tag{1}
$$

Task #1

Implement the above model in your favourite environment using the following parameter set: $g_L = 2$, $V_L = -60$, $g_{Ca} = 4$, $V_{Ca} = 120$, $g_K = 8$, $V_K = -84$, $C = 20$, $I = 0$, $V_1 = -1.2$, $V_2 = 18$, $V_3 = 2$, $V_4 = 30$, $\phi = 0.04$. Time is measured in milliseconds in the model, so make the total time for the simulation equal 100 ms.

Simulate the system for the initial conditions $w(0) = 0.014873$ and $V(0) = -60.899$. Now change the conditions to $V(0) = -20$ leaving $w(0)$ as is. Finally, try $V(0) = -10$. What have you observed for these three initial conditions? Does the dynamics change qualitatively? Plot the kinetics of the Ca^{2+} and K^+ currents which are $I_{Ca} = g_{Ca}m_\infty(V)(V - V_{Ca})$ and $I_K = g_K w(V - V_K)$.

Given the above three simulation experiments can you explain what the phenomenon of *excitability* means? Plot the phase portraits for the above simulations and note the difference between the three cases (Hint: use total time of simulation equal 250 ms when plotting phase portraits).

The Morris-Lecar model is particularly useful since allows for the dynamical investigation on the 2D phase portrait. As you remember Hodgkin-Huxley model is 4D.

Task #2

Recall the notion of (major) nullclines. There are two major nullclines, which are curves where $dV/dt = 0$ and $dw/dt = 0$. Draw the major nullclines for the Morris-Lecar model.

*This project work is inspired by the Bard Ermentrout's XPPAUT tutorial: `http://www.math.pitt.edu/~bard/bardware/tut/xpptut3.html`

[1] `http://www.scholarpedia.org/article/Morris-Lecar_model`

Recall also that the intersection points of the major nullclines are the fixed point attractors of the system. Try to find the steady states both using data browser of the curves you drew and by any numerical procedure (Hint: use `fsolve` in Matlab).

Recall the matrix `A` of the variable coefficients of the linearized system (it is frequently called `Jacobian`), which we used to analyze the stability of steady states. Try to find numerically the Jacobian matrix and evaluate it at the steady state(s) (Hint: in Matlab one can use `gradient` function along with `subs` to instantiate the symbolic expressions with real values).

Finally, using the Jacobian matrix try to find, also numerically, its eigenvalues ($\lambda_{1,2}$) for the given steady state(s). What can you say about steady state's stability? What is the type of the steady state(s)?

Task #3

Set $I = 120$ and simulate for 250 ms or larger. What do you see? Try to determine the steady state stability using procedures from the previous task given these new conditions. What is the stability of the steady state?

Compare the eigenvalues for $I = 0$ (previous task) and for $I = 120$. What can you say about the dynamical transition? Does any bifurcation take place when the current I changes? What is the bifurcation, if any?

Set $I = 80$ and simulate again. What has changed in the steady state as compared to the $I = 0$ case? Try to identify value of the current I at which the steady state goes from *stable* to *unstable*.

Task #3a (Advanced)

If you are familiar with the bifurcation analysis tools (e.g. AUTO, MATCONT, CONTENT etc.), try to perform the continuation of the steady state at $I = 0$ up to $I = 250$ (or so). Look for bifurcations occurring while the continuation procedure goes. Expand all bifurcations to see what novel dynamical behaviors they give rise to.

If you are not familiar with any bifurcation analysis toolbox, just jump to the next task.

Task #4

Let us change the parameter a bit as compared to the default set: $V_3 = 12$, $V_4 = 17.4$, and $\phi = 0.066666667$.

Look at the nullclines for $I = 0$, $I = 30$, $I = 50$. Determine the stability of all fixed points you see (Hint: use different initial guesses for the `fsolve` in Matlab to study each steady state separately). For $I = 30$ what is the type of the middle intersection point of the nullclines? What function does this steady state serve in the system with the given parameter set? What are the stable attractors of the system and with what fixed points they are "associated" with?

Set $\phi = 0.25$ and repeat the exercise from the beginning. What are the stable attractors now? Does the system possess any periodic behavior?